Spectral heat content on a class of fractal sets for subordinate killed Brownian motions

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Abstract

We study the spectral heat content for a class of open sets with fractal boundaries determined by similitudes in $\mathbb{R}^d$, $d \geq 1$, with respect to subordinate killed Brownian motions via $\alpha/2$-stable subordinators and establish the asymptotic behavior of the spectral heat content as $t \to 0$ for the full range of $\alpha \in (0, 2)$. Our main theorems show that these asymptotic behaviors depend on whether the sequence of logarithms of the coefficients of the similitudes is arithmetic when $\alpha \in [d - \mathfrak{b}, 2)$, where $\mathfrak{b}$ is the interior Minkowski dimension of the boundary of the open set. The main tools for proving the theorems are the previous results on the spectral heat content for Brownian motions and the renewal theorem.

1 Introduction

The spectral heat content on an open set $D \subset \mathbb{R}^d$ measures the total heat that remains on D at time $t > 0$ when the initial temperature is one with Dirichlet boundary condition outside D. The spectral heat content with respect to Brownian motion has been studied intensively, not only for domains with smooth boundary (for example, see [5]) but also for certain domains with fractal boundaries such as the s-adic von Koch snowflake (see [3, 4, 6, 7]).

Recently, there have been increasing interests in the spectral heat content for more general Lévy processes (see [2, 8, 10, 14]). In [8] the authors studied the spectral heat content for some class of Lévy processes on bounded open sets in $\mathbb{R}$. In particular, it is proved in [8] (see Theorems 4.2 and 4.14) that when the underlying open set has infinitely many components (or infinitely many non-adjacent components in the case of the Cauchy process) the decay rate of the spectral heat content is strictly bigger than that of the spectral heat content with respect to open sets with finitely many components. Hence, a natural question is to determine the exact decay rate of the spectral heat content for Lévy processes when there are infinitely many components in $D \subset \mathbb{R}$.

Many Lévy processes can be realized as subordinate (time-changed) Brownian motions, where the time change is given by an independent subordinator. Since we need two operations to define

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the spectral heat content for subordinate Brownian motions, time-change and killing, there are two objects that can be called the spectral heat content for subordinate Brownian motions: One is related to the *killed subordinate Brownian motions* (do time-change first then kill the time-changed Brownian motions when they exit the domain under consideration) and the other is related to *subordinate killed Brownian motions* (kill Brownian motions when they exit the domain, then make time-change for the killed Brownian motions). Even though the spectral heat content for killed subordinate Brownian motions is a natural object to study as it covers a large class of the spectral heat content for killed Lévy processes, the spectral heat content for subordinate killed Brownian motions is also important as it oftentimes gives useful information on the spectral heat content for the killed subordinate Brownian motions. For example, in [15] the asymptotic behavior of the spectral heat content for subordinate killed Brownian motions with respect to stable subordinators provides crucial information on the spectral heat content for killed stable processes.

In this paper, we study the spectral heat content for the subordinate killed Brownian motions (see (2.10) below for the definition) when the underlying subordinator is a stable subordinator $S^{(\alpha/2)} = \{S_t^{(\alpha/2)}\}_{t\geq 0}$ whose Laplace transform is given by

$$\mathbb{E}[e^{-\lambda S_t^{(\alpha/2)}}] = e^{-t\lambda^{\alpha/2}}, \quad \lambda > 0, \ \alpha \in (0,2), \tag{1.1}$$

and the underlying open sets have fractal boundaries which are determined by similitudes in $\mathbb{R}^d$. Our main results answer the aforementioned question and they show that the exact decay rates of the spectral heat content depend on whether the sequence of logarithms of the coefficients of the similitudes is arithmetic when $\alpha \in [d-\mathfrak{b}, 2)$, where $\mathfrak{b} \in (d-1, d)$ is the interior Minkowski dimension of the boundary of the underlying set (cf. (2.1) below for definition).

The main technique for studying the small time asymptotic behavior of the spectral heat content for open sets with fractal boundaries in this paper is *the renewal theorem* in [13]. Two crucial properties that we need are the additivity property of the spectral heat content under disjoint union (Lemma 2.5) and the scaling property of the subordinate killed Brownian motions with respect to stable subordinators (Lemma 2.7). However, in order to apply the renewal theorem, it is necessary to have an exponential decay condition (see (2.9)) which is only valid when $\alpha \in (d-\mathfrak{b}, 2)$. In order to establish the asymptotic behavior of the spectral heat content as $t \to 0$ for the full range of $\alpha \in (0,2)$, we will use the weak convergence of Lévy measure (see Proposition 3.7) to establish the asymptotic behavior for the case when $\alpha \in (0, d-\mathfrak{b})$. The result is given as Theorem 3.8. It is noteworthy to observe that when $\alpha \in (0, d-\mathfrak{b})$ the theorem is independent of whether the sequence $\{\ln(1/r_j)\}$ of the logarithms of the coefficients of the similitudes is arithmetic or not. The remaining case when $\alpha = d-\mathfrak{b} \in (0,1)$ is proved in Theorems 3.5 and 3.6 for the arithmetic and non-arithmetic cases, respectively. We observe that there is an extra logarithm term $\ln(1/t)$ in the decay rate of the spectral heat content when $\alpha = d-\mathfrak{b}$. This is due to the fact that the heat loss $|G| - Q_G^{(2)}(u)$ for Brownian motions on the open set G with fractal boundaries, where $|G| - Q_G^{(2)}(u) = \int_G \mathbb{P}_x(\tau_G^{(2)} \leq u)dx$ and $\tau_G^{(2)}$ is the first exit time of the Brownian motions out of

G, is barely non-integrable with respect to the law of stable subordinator $S_t^{(\alpha/2)}$. Note that this occurrence of extra logarithm term $\ln(1/t)$ is also observed for smooth open sets in [1, 14, 15], but it happens at a different index α. More specifically, for the spectral heat content on smooth sets this phenomenon happens when $\alpha = 1$, whereas for open sets with fractal boundaries as in our case, this happens when $\alpha = d - \mathfrak{b}$, which is strictly less than 1.

The organization of this paper is as follows. In Section 2 we set up notations, define the class of open sets with fractal boundaries in (2.3), and recall some facts which will be used for proving our main theorems. In particular, we recall the renewal theorem in [13] and the result for the spectral heat content for Brownian motions in Theorem 2.4. Section 3 is the main part of this paper and here we study the spectral heat content for subordinate killed Brownian motions. The main results are Theorems 3.3, 3.5, 3.6, and 3.8. This section is divided into three subsections for the cases $\alpha \in (d-\mathfrak{b}, 2)$, $\alpha = d - \mathfrak{b}$, and $\alpha \in (0, d-\mathfrak{b})$, respectively.

We use c_i to denote constants whose values are unimportant and may change from one appearance to another. The notations $\mathbb{P}_x$ and $\mathbb{E}_x$ mean probability and expectation of the underlying processes started at $x \in \mathbb{R}^d$, and we use $\mathbb{P} = \mathbb{P}_0$ and $\mathbb{E} = \mathbb{E}_0$ to simplify notations.

2 Preliminaries

In this section, we introduce notations and recall some facts that will be used for proving the main theorems in Section 3.

2.1 Some geometric notions

We first recall some geometric notions and the definition of the class of open sets with fractal boundaries from [13]. See also [11, 12] for more recent developments.

For any bounded (open) set $G \subset \mathbb{R}^d$ with boundary ∂G and any $\varepsilon > 0$, let

$$G_\varepsilon^{\text{int}} = \{x \in G : \text{dist}(x, \partial G) < \varepsilon\}$$

be the interior Minkowski sausage of radius ε of the boundary ∂G. We denote by $\mu(\varepsilon; G)$ the d-dimensional Lebesgue measure of $G_\varepsilon^{\text{int}}$. For any $s > 0$, define

$$\mathscr{M}^*(s, \partial G) = \limsup_{\varepsilon \to 0} \varepsilon^{-(d-s)} \mu(\varepsilon; G)$$

and

$$\mathscr{M}_*(s, \partial G) = \liminf_{\varepsilon \to 0} \varepsilon^{-(d-s)} \mu(\varepsilon; G).$$

Following [13, Definition 1.2], the interior Minkowski dimension of ∂G (which is also called the Minkowski dimension of ∂G relative to G) is defined by

$$\dim_{\text{M}}^{\text{int}}(\partial G) = \inf\{s > 0 : \mathscr{M}^*(s, \partial G) = 0\} = \sup\{s > 0 : \mathscr{M}_*(s, \partial G) = \infty\}. \tag{2.1}$$

If, for $s = \dim_{\mathrm{M}}^{\mathrm{int}}(\partial G)$, we have $0 < \mathscr{M}^*(s, \partial G) = \mathscr{M}_*(s, \partial G) < \infty$, then, ∂G is said to be Minkowski measurable relative to G.

Definition 2.1 *A map $R : \mathbb{R}^d \to \mathbb{R}^d$ is called a similitude with coefficient $r > 0$ if*

$$|Rx - Ry| = r|x - y| \quad \text{for all } x,\, y \in \mathbb{R}^d.$$

It is well known (cf. e.g., [9, (1) Proposition] or [13, p.191]) that any similitude is a composition of a homothety with coefficient r, an orthonormal transform, and a translation.

Now we define the class of open sets with fractal boundaries that we will consider in this paper. Let $G_0 \subset \mathbb{R}^d$ be a bounded open set. When $d = 1$, we assume that G_0 is a bounded open interval whereas when $d \geq 2$, we assume that G_0 is assumed to be a bounded $C^{1,1}$ open set. Let R_j $(1 \leq j \leq N)$ be similitudes with coefficients r_j, respectively.

For each $n \geq 1$, define $\Upsilon_n = \{\boldsymbol{j} = (j_1, \ldots, j_n),\ 1 \leq j_i \leq N\}$. We define the set G by

$$G = \left(\bigcup_{n=1}^{\infty} \bigcup_{\boldsymbol{j} \in \Upsilon_n} \mathscr{R}_{\boldsymbol{j}} G_0 \right) \cup G_0, \tag{2.2}$$

where, for every $\boldsymbol{j} = (j_1, \ldots, j_n) \in \Upsilon_n$, $\mathscr{R}_{\boldsymbol{j}}$ is the similitude defined by $\mathscr{R}_{\boldsymbol{j}} = R_{j_1} \circ \cdots \circ R_{j_n}$. It follows from (2.2) (see also [13, Equation (1.8)]) that G can be represented as

$$G = \left(\bigcup_{j=1}^{N} R_j G \right) \cup G_0. \tag{2.3}$$

We assume that all the sets $R_j G$, $1 \leq j \leq N$, and G_0 in (2.3) are pairwise disjoint. As in [13, Equation (1.7)] we also assume that $\sum_{j=1}^{N} r_j^d < 1 < \sum_{j=1}^{N} r_j^{d-1}$. Since the expressions in (2.3) are pairwise disjoint, we have

$$|G| = \sum_{j=1}^{N} r_j^d |G| + |G_0|. \tag{2.4}$$

Also, the condition $\sum_{j=1}^{N} r_j^d < 1 < \sum_{j=1}^{N} r_j^{d-1}$ ensures that G has a finite volume and there exists a unique number $\mathfrak{b} \in (d-1, d)$ such that

$$\sum_{j=1}^{N} r_j^{\mathfrak{b}} = 1. \tag{2.5}$$

It follows from [13, Theorem A] that the number $\mathfrak{b}$ is equal to the interior Minkowski dimension of ∂G. For more details, see [13] (pages 193–194).

As an illustration, we consider the following examples. Let R_1 and R_2 be two similitudes on $\mathbb{R}$ defined by

$$R_1(x) = \frac{1}{3}x \quad \text{and} \quad R_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

We take $G_0 = (\frac{1}{3}, \frac{2}{3})$. Then, it is easy to observe that the set defined in (2.2) is given by $G = (0,1)\backslash\mathfrak{C}$, where $\mathfrak{C}$ is the standard ternary Cantor set and G satisfies (2.3), and $\mathfrak{b} = \log 2/\log 3$. Similarly, the open set $G \subset (0,\infty)^2$ with the Sierpínski gasket as its boundary can be obtained from G_0 being the open triangle with vertices $(1/4, \sqrt{3}/4)$, $(1/2, 0)$ and $(3/4, \sqrt{3}/4)$ (notice that the boundary ∂G_0 is not $C^{1,1}$) and three similitudes on $\mathbb{R}^2$ defined by

$$R_1(x) = \frac{1}{2}x, \quad R_2(x) = \frac{1}{2}x + \Big(\frac{1}{4}, \frac{\sqrt{3}}{4}\Big) \quad \text{and} \quad R_3(x) = \frac{1}{2}x + \Big(\frac{1}{2}, 0\Big).$$

The (interior) Minkowski dimension of the boundary ∂G is $\mathfrak{b} = \log 3/\log 2$.

2.2 The renewal theorem

Now we state a version of the renewal theorem from [13]. Let $f : \mathbb{R} \to \mathbb{R}$ be a map. For any $\gamma \in \mathbb{R}$ define

$$L_\gamma f(z) = f(z - \gamma),$$

and

$$Lf(z) = \sum_{j=1}^{N} c_j L_{\gamma_j} f(z) = \sum_{j=1}^{N} c_j f(z - \gamma_j),$$

where $c_j > 0$, γ_j are distinct points in $\mathbb{R}$ with $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_N$, and $\sum_{j=1}^{N} c_j = 1$.

Consider the following *renewal equation*

$$f = Lf + \phi. \tag{2.6}$$

The question is to find a function f for a given function ϕ. Intuitively, it is natural to expect that the solution of the renewal equation is given by

$$f(z) = \sum_{n=0}^{\infty} L^n \phi(z) = \phi(z) + \sum_{n=1}^{\infty} \sum_{c_{i_1}, \cdots, c_{i_n}} c_{i_1} \cdots c_{i_n} L_{\gamma_{i_1}} \cdots L_{\gamma_{i_n}} \phi(z). \tag{2.7}$$

The following renewal theorem says it is indeed the case under certain conditions. We say a set of finite real numbers $\{\gamma_1, \cdots, \gamma_N\}$ is *arithmetic* if $\frac{\gamma_i}{\gamma_j} \in \mathbb{Q}$ for all indices. The maximal number γ such that $\frac{\gamma_i}{\gamma} \in \mathbb{Z}$ is called the span of $\{\gamma_1, \cdots, \gamma_N\}$. If the set is not arithmetic, it is called *non-arithmetic.*

Theorem 2.2 (Renewal Theorem [13]) *Suppose that a map $f : \mathbb{R} \to \mathbb{R}$ satisfies the renewal equation* (2.6) *and it satisfies*

$$\lim_{z \to -\infty} f(z) = 0, \tag{2.8}$$

and ϕ is a piecewise continuous function on $\mathbb{R}$ satisfying the condition

$$|\phi(z)| \le c_1 e^{-c_2|z|}, \quad z \in \mathbb{R}, \tag{2.9}$$

for some constants $c_1, c_2 > 0$. Then, the solution of the renewal equation (2.6) is given by (2.7). Furthermore, if $\{\gamma_j\}$ is non-arithmetic, then

$$f(z) = \frac{1}{\sum_{j=1}^{N} c_j\gamma_j} \int_{-\infty}^{\infty} \phi(x)dx + o(1), \;\; \textit{as } z \to \infty.$$

If $\{\gamma_j\}$ is arithmetic with span γ, then

$$f(z) = \frac{\gamma}{\sum_{j=1}^{N} c_j\gamma_j} \sum_{k=-\infty}^{\infty} \phi(z - k\gamma) + o(1), \;\; \textit{as } z \to \infty.$$

2.3 The spectral heat content of subordinate killed Brownian motions

For an open set $D \subset \mathbb{R}^d$ we define the spectral heat content for Brownian motion $W = \{W_t\}_{t\geq 0}$ on $\mathbb{R}^d$ as

$$Q_D^{(2)}(t) = \int_D \mathbb{P}_x\big(\tau_D^{(2)} > t\big)dx, \quad \tau_D^{(2)} = \inf\{t > 0 : W_t \notin D\}.$$

We record the following lemma from [13, Lemma 4.4]. Note that there was a typo there and r^2 in (4.7) should be written as r^d.

Lemma 2.3 (Lemma 4.4 [13]) *Let D be an open set in $\mathbb{R}^d$ and R be a similitude with coefficient $r > 0$. Then,*

$$Q_{RD}^{(2)}(t) = r^d Q_D^{(2)}(t/r^2).$$

Proof. In [13], it is stated that this lemma can be proved by change of variables. For convenience of readers, we provide a probabilistic proof. We first show that

$$\tau_{RD}^{(2)} \text{ under } \mathbb{P}_{Rx} \text{ is equal in distribution to } r^2\tau_D^{(2)} \text{ under } \mathbb{P}_x.$$

Recall that any similitude R is a composition of a homothety with coefficient r, an orthonormal transform, and a translation, so that we write R as $R = S_rT_aO$, where $S_r(x) = rx$, $T_a(x) = x - a$, and O is an orthonormal transformation in $\mathbb{R}^d$ with $x, a \in \mathbb{R}^d$. By the scaling property of Brownian motion $W = \{W_t\}_{t\geq 0}$ we have $W_t \stackrel{d}{=} S_rW_{tr^{-2}}$. Hence, by letting $u = tr^{-2}$

$$\begin{aligned}\tau_{RD}^{(2)} &= \inf\{t > 0 | W_t \notin RD\} = \inf\{t > 0 | R^{-1}W_t \notin D\} \text{ under } \mathbb{P}_{Rx} \\ &\stackrel{d}{=} \inf\{t |\, R^{-1}S_rW_{tr^{-2}} \notin D\} = r^2 \inf\{u > 0 |\, R^{-1}S_rW_u \notin D\} \text{ under } \mathbb{P}_{S_{1/r}Rx} = \mathbb{P}_{Ox-a}.\end{aligned}$$

It is elementary that $R^{-1} = S_{1/r}T_{-O^{-1}ra}O^{-1}$ and $R^{-1}S_r = S_{1/r}T_{-O^{-1}ra}O^{-1}S_r = S_{1/r}T_{-O^{-1}ra}S_rO^{-1} = S_{1/r}S_rT_{-O^{-1}a}O^{-1} = T_{-O^{-1}a}O^{-1}$. Observe that under the law $\mathbb{P}_{Ox-a}$, $T_{-O^{-1}a}O^{-1}W_u \notin D$ if and only if $W_u \notin OD - a$, which in turn is equivalent to $W_u \notin OD$ under $\mathbb{P}_{Ox}$. Hence, due to the rotation invariance of W, $\tau_{RD}^{(2)}$ under $\mathbb{P}_{Rx}$ is equal in law to $r^2\tau_D^{(2)}$ under $\mathbb{P}_x$.

Now by the change of variable $x = Ry$

$$Q_{RD}^{(2)}(t) = \int_{RD} \mathbb{P}_x(\tau_{RD}^{(2)} > t)dx = \int_D \mathbb{P}_{Ry}(\tau_{RD}^{(2)} > t)r^d dy = r^d \int_D \mathbb{P}_y(r^2\tau_D^{(2)} > t)dy = r^d Q_D^{(2)}(t/r^2).$$

□

We recall the following theorem for the spectral heat content for Brownian motion from [13, Theorem D]. Note that we found several typos in the statement of [13, Theorem D]. Readers who are interested in the proof could consult the proof of Theorem 3.3 with $\alpha = 2$.

Theorem 2.4 (Theorem D [13]) *Let G be a set defined as in (2.2) with R_j being similitude with coefficient r_j, and G_0 is either a bounded open interval when $d = 1$, or a bounded $C^{1,1}$ open set when $d \geq 2$.*

1. *If $\{\ln(\frac{1}{r_j})\}_{j=1}^{N}$ is non-arithmetic, then as $t \to 0$,*

$$Q_G^{(2)}(t) = |G| - Ct^{\frac{d-\mathfrak{b}}{2}} + o(t^{\frac{d-\mathfrak{b}}{2}}), \quad \text{where } C = \frac{\int_0^\infty \left(|G_0| - Q_{G_0}^{(2)}(u)\right) u^{-(1+\frac{d-\mathfrak{b}}{2})} du}{\sum_{j=1}^{N} r_j^{\mathfrak{b}} \ln(\frac{1}{r_j^2})}.$$

2. *If $\{\ln(\frac{1}{r_j})\}_{j=1}^{N}$ is arithmetic with span ρ, then as $t \to 0$,*

$$Q_G^{(2)}(t) = |G| - s(-\ln t) t^{\frac{d-\mathfrak{b}}{2}} + o(t^{\frac{d-\mathfrak{b}}{2}}),$$

where the function $s(\cdot)$ is defined by

$$s(z) \quad := \quad \frac{2\rho}{\sum_{j=1}^{N} (r_j)^{\mathfrak{b}} \ln(\frac{1}{r_j^2})} \sum_{n=-\infty}^{\infty} \left(|G_0| - Q_{G_0}^{(2)}\big(e^{-(z-2n\rho)}\big)\right) e^{\frac{d-\mathfrak{b}}{2}(z-2n\rho)}.$$

Now we introduce the spectral heat content for subordinate killed Brownian motions. Let $W = \{W_t\}_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^d$ and let $S^{(\alpha/2)} = \{S_t^{(\alpha/2)}\}_{t\geq 0}$ be an $(\alpha/2)$-stable subordinator with Laplace transform given by (1.1), which is independent of W. Let D be any open set in $\mathbb{R}^d$. Then, the spectral heat content $\tilde{Q}_D^{(\alpha)}(t)$ for subordinate killed Brownian motions with respect to stable subordinator $S^{(\alpha/2)}$ on D is defined by

$$\tilde{Q}_D^{(\alpha)}(t) = \int_D \mathbb{P}_x\Big(\tau_D^{(2)} > S_t^{(\alpha/2)}\Big) dx, \tag{2.10}$$

where $\tau_D^{(2)} = \inf\{t > 0 : W_t \notin D\}$.

We will need the following important properties of $\tilde{Q}_D^{(\alpha)}(t)$: One is the additivity under disjoint union and another is the scaling property.

Lemma 2.5 *Let D_1, D_2 be open sets in $\mathbb{R}^d$ with $D_1 \cap D_2 = \emptyset$. Then, for $\alpha \in (0,2)$*

$$\tilde{Q}_{D_1 \cup D_2}^{(\alpha)}(t) = \tilde{Q}_{D_1}^{(\alpha)}(t) + \tilde{Q}_{D_2}^{(\alpha)}(t).$$

Proof. Note that

$$\tilde{Q}^{(\alpha)}_{D_1\cup D_2}(t) = \int_{D_1\cup D_2} \mathbb{P}_x\Big(\tau^{(2)}_{D_1\cup D_2} > S^{(\alpha/2)}_t\Big)dx$$
$$= \int_{D_1} \mathbb{P}_x\Big(\tau^{(2)}_{D_1\cup D_2} > S^{(\alpha/2)}_t\Big)dx + \int_{D_2} \mathbb{P}_x\Big(\tau^{(2)}_{D_1\cup D_2} > S^{(\alpha/2)}_t\Big)dx.$$

Note that under $\mathbb{P}_x$ with $x \in D_1$, the continuity of the sample paths of $W = \{W_t\}_{t\geq 0}$ implies that

$$\tau^{(2)}_{D_1\cup D_2} = \inf\{t > 0 : W_t \notin D_1 \cup D_2\} = \inf\{t > 0 : W_t \notin D_1\} = \tau^{(2)}_{D_1}.$$

Hence, we have

$$\int_{D_1} \mathbb{P}_x\Big(\tau^{(2)}_{D_1\cup D_2} > S^{(\alpha/2)}_t\Big)dx = \int_{D_1} \mathbb{P}_x\Big(\tau^{(2)}_{D_1} > S^{(\alpha/2)}_t\Big)dx = \tilde{Q}^{(\alpha)}_{D_1}(t).$$

It can be proved in the same way that the integral on D_2 gives $\tilde{Q}^{(\alpha)}_{D_2}(t)$. □

Remark 2.6 *Let $Q^{(\alpha)}_D(t) := \int_D \mathbb{P}_x\big(\tau^{(\alpha)}_D > t\big)dx$ be the spectral heat content for killed stable processes, where $\tau^{(\alpha)}_D$ is the first exit time of the α-stable process $W_{S^{(\alpha/2)}} = \{W_{S^{(\alpha/2)}_t}\}_{t\geq 0}$. This is the spectral heat content related to the killed subordinate Brownian motions by stable subordinators. Note that for disjoint sets D_1 and D_2 we have*

$$Q^{(\alpha)}_{D_1\cup D_2}(t) = \int_{D_1\cup D_2} \mathbb{P}\Big(\tau^{(\alpha)}_{D_1\cup D_2} > t\Big)dx = \int_{D_1} \mathbb{P}\Big(\tau^{(\alpha)}_{D_1\cup D_2} > t\Big)dx + \int_{D_2} \mathbb{P}\Big(\tau^{(\alpha)}_{D_1\cup D_2} > t\Big)dx$$
$$\geq \int_{D_1} \mathbb{P}\Big(\tau^{(\alpha)}_{D_1} > t\Big)dx + \int_{D_2} \mathbb{P}\Big(\tau^{(\alpha)}_{D_2} > t\Big)dx = Q^{(\alpha)}_{D_1}(t) + Q^{(\alpha)}_{D_2}(t).$$

Furthermore, the inequality can be strict as $\tau^{(\alpha)}_{D_1\cup D_2} \neq \tau^{(\alpha)}_{D_1}$ when the process starts at $x \in D_1$ because the process starting at $x \in D_1$ can jump into D_2 without visiting the complement of $D_1\cup D_2$. Hence, the spectral heat content for killed subordinate Brownian motions does not satisfy the additivity property under disjoint union.

Lemma 2.7 *Let R be a similitude with coefficient r and D is any open set in $\mathbb{R}^d$. Then, we have*

$$\tilde{Q}^{(\alpha)}_{RD}(t) = r^d \tilde{Q}^{(\alpha)}_D(t/r^\alpha), \quad t > 0.$$

Proof. From the proof of Lemma 2.3, $\tau^{(2)}_{RD}$ under $\mathbb{P}_{Rx}$ is equal in distribution to $r^2\tau^{(2)}_D$ under $\mathbb{P}_x$. By the change of variable $x = Ry$ and the scaling property of $S^{(\alpha/2)}_t$ we have

$$\tilde{Q}^{(\alpha)}_{RD}(t) = \int_{RD} \mathbb{P}_x\Big(\tau^{(2)}_{RD} > S^{(\alpha/2)}_t\Big)dx = \int_D \mathbb{P}_{Ry}\Big(\tau^{(2)}_{RD} > S^{(\alpha/2)}_t\Big)r^d dy$$
$$= \int_D \mathbb{P}_y\Big(r^2\tau^{(2)}_D > S^{(\alpha/2)}_t\Big)r^d dy = \int_D \mathbb{P}_y\Big(\tau^{(2)}_D > r^{-2}S^{(\alpha/2)}_t\Big)r^d dy$$
$$= \int_D \mathbb{P}_y\Big(\tau^{(2)}_D > S^{(\alpha/2)}_{tr^{-\alpha}}\Big)r^d dy = r^d \tilde{Q}^{(\alpha)}_D(t/r^\alpha).$$

This proves the lemma. □

3 Asymptotic behavior of the spectral heat content

3.1 The case of $\alpha \in (d - \mathfrak{b}, 2)$.

Analogous to [13, Theorem D], we will prove that the spectral heat content $\tilde{Q}_G^{(\alpha)}(t)$ as defined in (2.10) has the form

$$\tilde{Q}_G^{(\alpha)}(t) = |G| - f(-\ln t)t^{\frac{d-\mathfrak{b}}{\alpha}} + o(t^{\frac{d-\mathfrak{b}}{\alpha}}),$$

when $\alpha \in (d - \mathfrak{b}, 2)$, where $\mathfrak{b}$ is the constant in (2.5). We start with the following lemma.

Lemma 3.1 *Assume that $\alpha \in (d - \mathfrak{b}, 2)$. Suppose that G_0 is an open interval when $d = 1$ or a bounded $C^{1,1}$ open set when $d \geq 2$. Define*

$$\psi(z) = \left(|G_0| - \tilde{Q}_{G_0}^{(\alpha)}(e^{-z})\right) e^{\frac{d-\mathfrak{b}}{\alpha} z}.$$

Then, there exists a constant $c = c(\alpha, \mathfrak{b}, d, |G_0|) > 0$ such that

$$0 \leq \psi(z) \leq ce^{-c|z|} \text{ for all } z \in \mathbb{R}.$$

Proof. We define

$$\phi(t) = \left(|G_0| - \tilde{Q}_{G_0}^{(\alpha)}(t)\right) t^{-\frac{d-\mathfrak{b}}{\alpha}}, \quad t > 0.$$

Notice that for $\phi(t) \geq 0$ for all $t > 0$ and $\psi(z) = \phi(e^{-z})$ for each $z \in \mathbb{R}$.

The case when $z \leq 0$ is easy since we have

$$0 \leq \phi(t) \leq |G_0| t^{-\frac{d-\mathfrak{b}}{\alpha}}$$

and this gives

$$\psi(z) = \phi(e^{-z}) \leq |G_0| e^{\frac{d-\mathfrak{b}}{\alpha} z} = |G_0| e^{-\frac{d-\mathfrak{b}}{\alpha}|z|}$$

for all $z \leq 0$ and $\alpha \in (0, 2)$.

Now we handle the case when $z > 0$, or $t = e^{-z} \in (0, 1)$. First, assume that $\alpha \in (1, 2)$. Since G_0 is an interval when $d = 1$ or a bounded $C^{1,1}$ open set when $d \geq 2$, it follows from [14, Theorem 1.1] that there exists a constant $c_1 > 0$ such that

$$|G_0| - \tilde{Q}_{G_0}^{(\alpha)}(t) \leq c_1 t^{1/\alpha}$$

for all $0 < t \leq 1$. Hence,

$$\phi(t) \leq c_1 t^{\frac{(\mathfrak{b}+1)-d}{\alpha}} \text{ for } 0 < t \leq 1. \tag{3.1}$$

Since $\mathfrak{b} \in (d - 1, d)$ we note that $\frac{(\mathfrak{b}+1)-d}{\alpha} > 0$. We conclude that for all $z > 0$, by applying (3.1) with $t = e^{-z}$,

$$\psi(z) = \phi(e^{-z}) \leq c_2 e^{-c_3|z|}, \quad z \in \mathbb{R},$$

where $c_2 = \max(c_1, |G_0|)$ and $c_3 = \min(\frac{\mathfrak{b}+1-d}{\alpha}, \frac{d-\mathfrak{b}}{\alpha}) > 0$.

Second, when $\alpha = 1$ we have from [14, Theorem 1.1]

$$|G_0| - \tilde{Q}^{(\alpha)}_{G_0}(t) \le c_2 t \ln(1/t)$$

for all $0 < t \le 1$. Hence,

$$\phi(t) \le c_2 t^{1-(d-\mathfrak{b})} \ln(1/t) = c_2 t^{(\mathfrak{b}+1)-d} \ln(1/t) \quad \text{for } 0 < t \le 1,$$

and this implies

$$\psi(z) = \phi(e^{-z}) \le c_2 z e^{-z(\mathfrak{b}+1-d)} \text{ for } z \ge 0.$$

Since $\mathfrak{b} + 1 - d > 0$ there exists c_4 and $\eta > 0$ such that

$$\psi(z) \le c_4 e^{-\eta z} \quad \text{for all } z \ge 0.$$

Finally, we handle the case when $\alpha \in (d - \mathfrak{b}, 1)$. From [14, Theorem 1.1] we have

$$|G_0| - \tilde{Q}^{(\alpha)}_{G_0}(t) \le c_5 t$$

for all $0 < t \le 1$, and this implies $\phi(t) \le c_5 t^{\frac{\alpha-(d-\mathfrak{b})}{\alpha}}$ for $t \le 1$, which in turn implies $\psi(z) \le c_5 e^{-\frac{\alpha-(d-\mathfrak{b})}{\alpha} z}$ for $z \ge 0$. □

We will make use of the following simple lemma on the continuity of the map $t \to \tilde{Q}_D(t)$ which is proved in [10, Lemma 3.11].

Lemma 3.2 *For any open set D with $|D| < \infty$, the map $t \to \tilde{Q}^{(\alpha)}_D(t)$ is continuous.*

Here is the main theorem for the case of $\alpha \in (d - \mathfrak{b}, 2)$.

Theorem 3.3 *Let $\alpha \in (d - \mathfrak{b}, 2)$, where $\mathfrak{b}$ is the constant in (2.5) and G is a set given in (2.2) with G_0 being an open interval when $d = 1$ or a bounded $C^{1,1}$ open set when $d \ge 2$. If $\{\ln \frac{1}{r_j}\}_{j=1}^N$ is non-arithmetic, then we have*

$$\tilde{Q}^{(\alpha)}_G(t) = |G| - C_1 t^{\frac{d-\mathfrak{b}}{\alpha}} + o(t^{\frac{d-\mathfrak{b}}{\alpha}}) \quad \text{as } t \to 0, \tag{3.2}$$

where

$$C_1 = \frac{\int_{-\infty}^{\infty} \left(|G_0| - \tilde{Q}^{(\alpha)}_{G_0}(e^{-z})\right) e^{\frac{(d-\mathfrak{b})z}{\alpha}} dz}{\sum_{j=1}^N r_j^{\mathfrak{b}} \ln(1/r_j^\alpha)} = \frac{\int_0^\infty \left(|G_0| - \tilde{Q}^{(\alpha)}_{G_0}(t)\right) t^{-1-\frac{d-\mathfrak{b}}{\alpha}} dt}{\sum_{j=1}^N r_j^{\mathfrak{b}} \ln(1/r_j^\alpha)}.$$

If $\{\ln \frac{1}{r_j}\}_{j=1}^N$ is arithmetic with span ρ, then we have

$$\tilde{Q}^{(\alpha)}_G(t) = |G| - f(-\ln t) t^{\frac{d-\mathfrak{b}}{\alpha}} + o(t^{\frac{d-\mathfrak{b}}{\alpha}}) \text{ as } t \to 0, \tag{3.3}$$

where

$$f(z) = \frac{\alpha\rho}{\sum_{j=1}^N r_j^{\mathfrak{b}} \ln(1/r_j^\alpha)} \sum_{n=-\infty}^{\infty} \left(|G_0| - \tilde{Q}^{(\alpha)}_{G_0}(e^{-(z-\alpha n\rho)})\right) e^{\frac{d-\mathfrak{b}}{\alpha}(z-\alpha n\rho)}.$$

Proof. We denote by $f(\cdot)$ the function on $\mathbb{R}$ such that

$$\tilde{Q}_G^{(\alpha)}(t) = |G| - f(-\ln t)t^{\frac{d-\mathfrak{b}}{\alpha}} \quad \text{for all } \ t > 0. \tag{3.4}$$

We will show that f satisfies the renewal equation (2.6) and the conditions in Renewal Theorem 2.2.

Since $G = \bigcup_{j=1}^N R_j G \cup G_0$ and all expressions are disjoint, it follows from Lemmas 2.5 and 2.7 that

$$\tilde{Q}_G^{(\alpha)}(t) = \tilde{Q}^{(\alpha)}_{\bigcup_{j=1}^N R_j G \cup G_0}(t) = \sum_{j=1}^N \tilde{Q}^{(\alpha)}_{R_j G}(t) + \tilde{Q}^{(\alpha)}_{G_0}(t) = \sum_{j=1}^N r_j^d \tilde{Q}_G^{(\alpha)}(t/r_j^\alpha) + \tilde{Q}^{(\alpha)}_{G_0}(t). \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} |G| - f(-\ln t)t^{\frac{d-\mathfrak{b}}{\alpha}} &= \sum_{j=1}^N r_j^d \left(|G| - f\left(-\ln\left(\frac{t}{r_j^\alpha}\right)\right)\left(\frac{t}{r_j^\alpha}\right)^{\frac{d-\mathfrak{b}}{\alpha}} \right) + \tilde{Q}^{(\alpha)}_{G_0}(t) \\ &= \sum_{j=1}^N r_j^d |G| - \sum_{j=1}^N r_j^{\mathfrak{b}} \cdot f\left(-\ln t - \ln\left(\frac{1}{r_j^\alpha}\right)\right) t^{\frac{d-\mathfrak{b}}{\alpha}} + \left(|G_0| - (|G_0| - \tilde{Q}^{(\alpha)}_{G_0}(t)) \right). \end{aligned}$$

By combining this with (2.4), we conclude that

$$f(-\ln t) = \sum_{j=1}^N r_j^{\mathfrak{b}} \cdot f\left(-\ln t - \ln\left(\frac{1}{r_j^\alpha}\right)\right) + \phi(t),$$

where

$$\phi(t) = \left(|G_0| - \tilde{Q}^{(\alpha)}_{G_0}(t) \right) t^{-\frac{d-\mathfrak{b}}{\alpha}}.$$

By changing the variable $z = -\ln t$, we have

$$f(z) = \sum_{j=1}^N r_j^{\mathfrak{b}} \cdot f\left(z - \ln\left(\frac{1}{r_j^\alpha}\right)\right) + \phi(e^{-z}). \tag{3.6}$$

Thus, the function f satisfies the renewal equation.

Note that from (3.4) and Lemma 3.1 (exponential decay of $\psi(z)$) we have

$$\lim_{z\to-\infty} f(z) = \lim_{t\to\infty} f(-\ln t) = \lim_{t\to\infty} \left(|G| - \tilde{Q}_G^{(\alpha)}(t) \right) t^{-\frac{d-\mathfrak{b}}{\alpha}} = 0,$$

and this shows that the condition (2.8) holds. It follows from Lemma 3.1 that for any $\alpha \in (d-\mathfrak{b}, 2)$ there exist two constants $c_1, c_2 > 0$ such that

$$\psi(z) = \phi(e^{-z}) \le c_1 e^{-c_2|z|} \quad \text{for all } z \in \mathbb{R},$$

and the condition (2.9) holds.

Therefore, by Renewal Theorem 2.2, we see that if $\{\ln \frac{1}{r_j}\}_{j=1}^N$ is non-arithmetic, then as $z \to \infty$,

$$f(z) = \frac{1}{\sum_{j=1}^N r_j^{\mathfrak{b}} \ln(1/r_j^\alpha)} \int_{-\infty}^{\infty} \left(|G_0| - \tilde{Q}_{G_0}^{(\alpha)}(e^{-z})\right) e^{\frac{(d-\mathfrak{b})z}{\alpha}} dz + o(1).$$

This, together with (3.4), implies (3.2).

On the other hand, if $\{\ln \frac{1}{r_j}\}_{j=1}^N$ is arithmetic with span ρ, then $\{\ln \frac{1}{r_j^\alpha}\}_{j=1}^N$ is arithmetic with span $\alpha\rho$. Therefore, (3.3) follows from the arithmetic part of Renewal Theorem 2.2 and $f(z)$ is a periodic function with period $\alpha\rho$. □

3.2 The case of $\alpha = d - \mathfrak{b}$.

In this subsection, we study the case when $\alpha = d - \mathfrak{b} \in (0, 1)$. We need a simple lemma which is similar to [14, Lemma 3.2]. The proof is essentially the same with obvious modifications and will be omitted.

Lemma 3.4 *For any $\delta > 0$ and $\alpha \in (0, 2)$, we have*

$$\lim_{t \to 0} \frac{\mathbb{E}\left[\left(S_1^{(\alpha/2)}\right)^{\alpha/2}, 0 < S_1^{(\alpha/2)} < \delta t^{-2/\alpha}\right]}{\ln(1/t)} = \frac{1}{\Gamma(1 - \frac{\alpha}{2})}.$$

Theorem 3.5 *Let $\alpha = d - \mathfrak{b} \in (0, 1)$, where $\mathfrak{b}$ is the constant in (2.5) and G is a set given as (2.2) with G_0 being an open interval when $d = 1$ or a bounded $C^{1,1}$ open set when $d \geq 2$. Assume that $\{\ln(1/r_j)\}_{j=1}^N$ is arithmetic with span ρ. Define $A = \sup_{z \in \mathbb{R}} s(z)$ and $B = \inf_{z \in \mathbb{R}} s(z)$, where $s(z)$ is from Theorem 2.4.*

(1) Let $g(t) := \int_0^{t^{-2/\alpha}} s(-\ln(t^{2/\alpha} v)) v^{\alpha/2} \mathbb{P}(S_1^{(\alpha/2)} \in dv)$. Then, we have

$$|G| - \tilde{Q}_G^{(\alpha)}(t) = t g(t) + o(t \ln(1/t)). \tag{3.7}$$

(2) We have

$$\limsup_{t \to 0} \frac{g(t)}{\ln(1/t)} = \frac{A}{\Gamma(1 - \frac{\alpha}{2})} \quad \text{and} \quad \liminf_{t \to 0} \frac{g(t)}{\ln(1/t)} = \frac{B}{\Gamma(1 - \frac{\alpha}{2})}. \tag{3.8}$$

Proof. Note that by the scaling property of $S_t^{(\alpha/2)}$ we have

$$\begin{aligned}
|G| - \tilde{Q}_G^{(\alpha)}(t) &= \int_0^{\infty} \left(|G| - Q_G^{(2)}(u)\right) \mathbb{P}(S_t^{(\alpha/2)} \in du) = \int_0^{\infty} \left(|G| - Q_G^{(2)}(t^{2/\alpha} v)\right) \mathbb{P}(S_1^{(\alpha/2)} \in dv) \\
&= \int_0^{t^{-2/\alpha}} \left(|G| - Q_G^{(2)}(t^{2/\alpha} v)\right) \mathbb{P}(S_1^{(\alpha/2)} \in dv) + \int_{t^{-2/\alpha}}^{\infty} \left(|G| - Q_G^{(2)}(t^{2/\alpha} v)\right) \mathbb{P}(S_1^{(\alpha/2)} \in dv) \\
&= \int_0^{t^{-2/\alpha}} \frac{|G| - Q_G^{(2)}(t^{2/\alpha} v)}{(t^{2/\alpha} v)^{\frac{d-\mathfrak{b}}{2}}} (t^{2/\alpha} v)^{\frac{d-\mathfrak{b}}{2}} \mathbb{P}(S_1^{(\alpha/2)} \in dv) + \int_{t^{-2/\alpha}}^{\infty} \left(|G| - Q_G^{(2)}(t^{2/\alpha} v)\right) \mathbb{P}(S_1^{(\alpha/2)} \in dv).
\end{aligned}$$

Hence, we have

$$|G| - \tilde{Q}_G^{(\alpha)}(t) - tg(t) = t\int_0^{t^{-2/\alpha}} \left(\frac{|G| - Q_G^{(2)}(t^{2/\alpha}v)}{(t^{2/\alpha}v)^{\frac{d-\mathfrak{b}}{2}}} - s(-\ln(t^{2/\alpha}v)) \right) v^{\frac{d-\mathfrak{b}}{2}} \mathbb{P}(S_1^{(\alpha/2)} \in dv)$$
$$+ \int_{t^{-2/\alpha}}^{\infty} \left(|G| - Q_G^{(2)}(t^{2/\alpha}v) \right) \mathbb{P}(S_1^{(\alpha/2)} \in du).$$

It follows from [14, Equation (2.8)] we have

$$\int_{t^{-2/\alpha}}^{\infty} \left(|G| - Q_G^{(2)}(t^{2/\alpha}v) \right) \mathbb{P}(S_1^{(\alpha/2)} \in du) \leq c\int_{t^{-2/\alpha}}^{\infty} |G| u^{-1-\frac{\alpha}{2}} du = o(t\ln(1/t)). \tag{3.9}$$

In this case, by applying Theorem 2.4 we have

$$\frac{|G| - Q_G^{(2)}(t^{2/\alpha}v)}{(t^{2/\alpha}v)^{\frac{d-\mathfrak{b}}{2}}} = s(-\ln(t^{2/\alpha}v)) + o(1) \quad \text{as } t \to 0. \tag{3.10}$$

From (3.10) for any $\varepsilon > 0$ there exists $t_0(\varepsilon)$ such that

$$\left| \frac{|G| - Q_G^{(2)}(t^{2/\alpha}v)}{(t^{2/\alpha}v)^{\frac{d-\mathfrak{b}}{2}}} - s(-\ln(t^{2/\alpha}v)) \right| < \varepsilon$$

for all $t \leq t_0$. Hence it follows from Lemma 3.4 that

$$\limsup_{t\to 0} \frac{t\int_0^{t^{-2/\alpha}} \left(\frac{|G| - Q_G^{(2)}(t^{2/\alpha}v)}{(t^{2/\alpha}v)^{\frac{d-\mathfrak{b}}{2}}} - s(-\ln(t^{2/\alpha}v)) \right) v^{\frac{d-\mathfrak{b}}{2}} \mathbb{P}(S_1^{(\alpha/2)} \in dv)}{t\ln(1/t)}$$
$$\leq \varepsilon \limsup_{t\to 0} \frac{\int_0^{t^{-2/\alpha}} v^{\alpha/2}\mathbb{P}(S_1^{(\alpha/2)} \in dv)}{\ln(1/t)} \leq \frac{\varepsilon}{\Gamma(1-\frac{\alpha}{2})}.$$

This establishes (3.7).

For (3.8), notice that

$$|G| - \tilde{Q}_G^{(\alpha)}(t) = \int_0^{t^{-2/\alpha}} \frac{|G| - Q_G^{(2)}(t^{2/\alpha}v)}{(t^{2/\alpha}v)^{\frac{d-\mathfrak{b}}{2}}} (t^{2/\alpha}v)^{\frac{d-\mathfrak{b}}{2}} \mathbb{P}(S_1^{(\alpha/2)} \in dv)$$
$$+ \int_{t^{-2/\alpha}}^{\infty} \left(|G| - Q_G^{(2)}(t^{2/\alpha}v) \right) \mathbb{P}(S_1^{(\alpha/2)} \in du).$$

As in (3.9) the second expression above is $o(t\ln(1/t))$ as $t \to 0$. For any $\varepsilon > 0$ it follows from (3.10) we have $\frac{|G| - Q_G^{(2)}(t^{2/\alpha}v)}{(t^{2/\alpha}v)^{\frac{d-\mathfrak{b}}{2}}} < A + \varepsilon$ for all sufficiently small t. This fact, together with (3.7) and Lemma 3.4 gives

$$\limsup_{t\to 0} \frac{g(t)}{\ln(1/t)} = \limsup_{t\to 0} \frac{|G| - \tilde{Q}_G^{(\alpha)}(t)}{t\ln(1/t)} \leq \frac{A+\varepsilon}{\Gamma(1-\frac{d-\mathfrak{b}}{2})}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\limsup_{t\to 0} \frac{g(t)}{\ln(1/t)} = \limsup_{t\to 0} \frac{|G| - \tilde{Q}_G^{(\alpha)}(t)}{t\ln(1/t)} \leq \frac{A}{\Gamma(1-\frac{d-\mathfrak{b}}{2})}. \tag{3.11}$$

For the lower bound, it follows from Theorem 2.4 that for any $\varepsilon > 0$ there exists a sequence $t_n \to 0$ such that

$$\frac{|G| - Q_G^{(2)}(t_n^{2/\alpha} v)}{(t_n^{2/\alpha} v)^{\frac{d-\mathfrak{b}}{2}}} \geq A - \varepsilon.$$

Hence, from Lemma 3.4, (3.9), and (3.11) we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{g(t_n)}{\ln(1/t_n)} &= \limsup_{n\to\infty} \frac{|G| - \tilde{Q}_G^{(\alpha)}(t_n)}{t_n \ln(1/t_n)} \\ &\geq (A-\varepsilon) \limsup_{n\to\infty} \int_0^{t_n^{-2/\alpha}} v^{\frac{\alpha}{2}} \mathbb{P}(S_1^{(\alpha/2)} \in dv) \geq \frac{A-\varepsilon}{\Gamma(1-\frac{d-\mathfrak{b}}{2})}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{t\to 0} \frac{g(t)}{\ln(1/t)} \geq \frac{A}{\Gamma(1-\frac{d-\mathfrak{b}}{2})}. \tag{3.12}$$

Hence, the lim sup version of (3.8) follows from (3.11) and (3.12), and the lim inf version can be proved in the same way. □

Here is the result for the non-arithmetic case. The proof is very similar to the proof of Theorem 3.5, hence it will be omitted.

Theorem 3.6 *Let $\alpha = d - \mathfrak{b} \in (0,1)$, where $\mathfrak{b}$ is the constant in (2.5) and G is a set given as (2.2) with G_0 being an open interval when $d = 1$ or a bounded $C^{1,1}$ open set when $d \geq 2$. Assume that $\{\ln(1/r_j)\}_{j=1}^N$ is non-arithmetic. Then, we have*

$$\lim_{t\to 0} \frac{|G| - \tilde{Q}_G^{(d-\mathfrak{b})}(t)}{t\ln(1/t)} = \frac{1}{2\Gamma(1-\frac{d-\mathfrak{b}}{2}) \sum_{j=1}^N r_j^{\mathfrak{b}} \ln(1/r_j)} \int_0^\infty \left(|G_0| - Q_{G_0}^{(2)}(u)\right) u^{-1-\frac{d-\mathfrak{b}}{2}} du.$$

3.3 The case of $\alpha \in (0, d-\mathfrak{b})$.

Now we handle the case when $\alpha \in (0, d-\mathfrak{b})$. The following proposition is proved in [10, Proposition 3.12].

Proposition 3.7 *Let f be a bounded continuous function on $(0,\infty)$ such that $\lim_{x\downarrow 0} \frac{f(x)}{x^\gamma}$ exists as a finite number for some constant $\gamma > \frac{\alpha}{2}$. Then, we have*

$$\lim_{t\downarrow 0} \int_0^\infty f(u) \frac{\mathbb{P}(S_t^{(\alpha/2)} \in du)}{t} = \frac{\alpha}{2\Gamma(1-\frac{\alpha}{2})} \int_0^\infty f(u)\, u^{-1-\frac{\alpha}{2}} du.$$

Theorem 3.8 *Let $\alpha \in (0, d-\mathfrak{b})$, where $\mathfrak{b}$ is the constant in (2.5) and G is a set given as (2.2) with G_0 being an open interval when $d = 1$ or a bounded $C^{1,1}$ open set when $d \geq 2$. Then, we have*

$$\lim_{t\to 0} \frac{|G| - \tilde{Q}_G^{(\alpha)}(t)}{t} = \frac{\alpha}{2\Gamma(1-\frac{\alpha}{2})} \int_0^\infty \left(|G| - Q_G^{(2)}(u)\right) u^{-1-\frac{\alpha}{2}} du.$$

Proof. Notice that we have

$$|G| - \tilde{Q}_G^{(\alpha)}(t) = \int_0^\infty \big(|G| - Q_G^{(2)}(u)\big)\mathbb{P}(S_t^{(\alpha/2)} \in du).$$

It follows from Theorem 2.4 that there exists constants c_1 such that

$$|G| - Q_G^{(2)}(u) \le c_1 u^{\frac{d-\mathfrak{b}}{2}} \text{ for } u \le 1.$$

Since $\alpha \in (0, d-\mathfrak{b})$, we can take $\gamma \in (\frac{\alpha}{2}, \frac{d-\mathfrak{b}}{2})$ and this implies

$$\lim_{u\to 0} \frac{|G| - Q_G^{(2)}(u)}{u^\gamma} = 0.$$

Now the conclusion of the theorem follows immediately from Proposition 3.7. □

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