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Higher Order Terms of the Spectral Heat Content for Killed Subordinate and Subordinate Killed Brownian Motions Related to Symmetric α-Stable Processes in $\mathbb{R}$

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Abstract

We investigate the 3rd term of the spectral heat content for killed subordinate and subordinate killed Brownian motions on a bounded open interval $D = (a, b)$ in a real line when the underlying subordinators are stable subordinators with index $\alpha \in (1, 2)$ or $\alpha = 1$. We prove that in the 3rd term of the spectral heat content, one can observe the length $b - a$ of the interval D.

Keywords Spectral heat content · Killed subordinate Brownian motions · Subordinate killed Brownian motions

Mathematics Subject Classification (2010) 60G51 · 60J76

1 Introduction

The classical spectral heat content $Q_D^{(2)}(t)$ measures the total heat that remains on a domain D with Dirichlet boundary condition and unit initial heat. The spectral heat content can be written in probabilistic terms, and it can be defined as

$$Q_D^{(2)}(t) = \int_D \mathbb{P}_x\left(\tau_D^{(2)} > t\right) dx,$$

where $\tau_D^{(2)} = \inf\{t > 0 : W_t \notin D\}$ is the first exit time from D by a Brownian motion $W = \{W_t\}_{t\geq 0}$. When the Brownian motion is replaced by other Lévy processes, the corresponding quantity is called the spectral heat content for the Lévy processes. It was recently studied intensively in [1, 2, 9].

One of the most commonly used jump type Lévy processes is the symmetric stable processes of index $\alpha \in (0, 2]$. When $\alpha = 2$, it is a Brownian motion whose sample paths are

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continuous with the characteristic exponent being $\mathbb{E}\left[e^{i\xi W_t}\right] = e^{-t\xi^2}$. When $\alpha \in (0, 2)$, they are pure-jump processes. Stable processes are in fact a special case of subordinate Brownian motions which are time-changed Brownian motions whose time change is given by stable subordinators $S_t^{(\alpha/2)}$ with Laplace exponent given by

$$\mathbb{E}\left[e^{-\lambda S_t^{(\alpha/2)}}\right] = e^{-t\lambda^{\alpha/2}}, \quad \lambda > 0.$$

When one studies the spectral heat content of subordinate Brownian motions, one needs to consider a time-change by a subordinator and killing the process when it first exits the domain under consideration. When we first do time-change and kill the processes, it is called *killed subordinate Brownian motions* and when we first kill the Brownian motions when they first exit the domain and do time-change into the killed Brownian motions, it is called *subordinate killed Brownian motions*. These two processes are closely related, and sometimes understanding the spectral heat content of one process helps understand the other. The spectral heat content for killed subordinate Brownian motions, when the subordinators are stable subordinators (killed stable processes), were studied in [1, 2], and the spectral heat content for subordinate killed Brownian motions were studied in [11]. In those papers, the authors found the asymptotic expansion of the spectral heat content up to the 2nd terms.

The purpose of this paper is to refine these results and find the 3rd terms of the spectral heat contents $\tilde{Q}_D^{(\alpha)}(t)$ and $Q_D^{(\alpha)}(t)$ for subordinate killed Brownian motions and killed subordinate Brownian motions, respectively, in a bounded open interval $D = (a, b) \subset \mathbb{R}$, when the subordinators are stable subordinators for $\alpha \in [1, 2)$. The main results of this paper are the followings. The explanation of notations of theorems will be postponed to Section 2 to introduce main results as quickly as possible. All asymptotic notations are as $t \downarrow 0$.

Theorem 1.1 *Let $D = (a, b) \subset \mathbb{R}$ with $a < b < \infty$, $|D| = b - a$, and $|\partial D| = 2$.*

(1) *Let $\alpha \in (1, 2)$. Then,*

$$|D| - Q_D^{(\alpha)}(t) = \mathbb{E}\left[\overline{X}_1^{(\alpha)}\right] |\partial D| t^{1/\alpha} - \frac{2^\alpha \Gamma\left(\frac{1+\alpha}{2}\right)}{(\alpha - 1)\pi^{1/2}\Gamma\left(1 - \frac{\alpha}{2}\right)|D|^{\alpha-1}} t + o(t). \quad (1.1)$$

(2) *Let $\alpha = 1$. Then,*

$$\begin{aligned} &|D| - Q_D^{(1)}(t) - \frac{1}{\pi}|\partial D| t \ln\left(\frac{1}{t}\right) \\ &= |\partial D| \left(\int_0^1 \mathbb{P}\left(\overline{X}_1^{(1)} > u\right) du + \frac{\ln |D|}{\pi} + \int_1^\infty \mathbb{P}\left(\overline{X}_1^{(1)} > u\right) - \frac{1}{\pi u} du\right) t + o(t). \quad (1.2) \end{aligned}$$

Theorem 1.2 *Let $D = (a, b) \subset \mathbb{R}$ with $a < b < \infty$, $|D| = b - a$, and $|\partial D| = 2$.*

(1) *Let $\alpha \in (1, 2)$. Then,*

$$|D| - \tilde{Q}_D^{(\alpha)}(t) = \mathbb{E}\left[\overline{W}_{S_1^{(\frac{\alpha}{2})}}\right] |\partial D| t^{1/\alpha} - \frac{2\alpha \int_0^\infty \mathbb{P}\left(\overline{W}_1 \geq u\right) u^{\alpha-1} du}{(\alpha - 1)\Gamma\left(1 - \frac{\alpha}{2}\right)|D|^{\alpha-1}} t + o(t). \quad (1.3)$$

(2) *Let $\alpha = 1$. Then,*

$$\begin{aligned} &|D| - \tilde{Q}_D^{(1)}(t) - \frac{2}{\pi}|\partial D| t \ln\left(\frac{1}{t}\right) \\ &= |\partial D| \left(\int_0^1 \mathbb{P}\left(\overline{W}_{S_1^{(\frac{1}{2})}} > u\right) du + \frac{2\ln |D|}{\pi} + \int_1^\infty \mathbb{P}\left(\overline{W}_{S_1^{(\frac{1}{2})}} > u\right) - \frac{2}{\pi u} du\right) t + o(t). \quad (1.4) \end{aligned}$$

Remark 1.3 When $\alpha \in (0,1)$, the asymptotic expansion for the spectral heat contents $Q_D^{(\alpha)}(t)$ or $\tilde{Q}_D^{(\alpha)}(t)$ are only known up to the second terms (see [2, Theorem 1.1] and [11, Theorem 1.1]). Asking the third terms when $\alpha \in (0,1)$ is definitely a very interesting question, and we intend to deal with this question in a future project.

Studying higher order terms is not only an interesting question in itself, but we could also observe that there are some different patterns in the asymptotic expansion of the spectral heat content for Brownian motions and other Lévy processes by studying higher order terms. For Brownian motions, it is well-known that for smooth domains D, the spectral heat content has the asymptotic expansion of the form $|D| - Q_D^{(2)}(t) \sim \sum_{n=1}^{\infty} a_n t^{\frac{n}{2}}$, where a_n has some geometric information about the domain D such as perimeter or mean curvature. Hence, it is natural to conjecture that at least when $\alpha \in (1,2)$, the spectral heat content for stable processes is of the form $|D| - Q_D^{(\alpha)}(t) \sim \sum_{n=1}^{\infty} b_n t^{\frac{n}{\alpha}}$. Theorems 1.1 and 1.2 say this is not the case and the asymptotic expansion involves terms that cannot be written as $t^{\frac{n}{\alpha}}$. Also, we observe that the 3rd term involves the length $b-a$ of the underlying interval $D=(a,b)$, hence one can determine the domain D uniquely up to locations, when D is a bounded open interval in $\mathbb{R}$ from the spectral heat content.

In this paper, we focus on the spectral heat content in dimension one. The geometry of open intervals in $\mathbb{R}$ is simple enough to allow to perform detailed computations, and this could be helpful to extend results of this paper into more general settings, such as the spectral heat content in higher dimensions or with respect to more general processes. These problems will be studied in forthcoming projects.

In order to prove the first part of Theorem 1.1 ($\alpha \in (1,2)$), we analyze the difference $|D| - Q_D^{(\alpha)}(t) - \mathbb{E}\left[\overline{X}_1^{(\alpha)}\right] |\partial D| t^{1/\alpha}$ directly and prove that it is of order t. Hence, the proof is quite straightforward in this case. For the second part of Theorem 1.1 ($\alpha = 1$), the computation becomes delicate because of the logarithmic term $t \ln(1/t)$. We utilize the exact form of the density of the supremum process $\overline{X}_t^{(1)} = \sup_{s \le t} X_s^{(1)}$ in [8] to compute the difference $\mathbb{P}\left(\overline{X}_1^{(1)} > u\right) - \frac{1}{\pi u}$ for large u, prove that main terms of order $t \ln(1/t)$ cancel out each other, and finally show that the remaining terms are of order t. In order to prove Theorem 1.2, we follow a similar path as Theorem 1.1. For the first part of Theorem 1.2 ($\alpha \in (1,2)$), we reprove [11, Theorem 1.1] when $D=(a,b)$ and $\alpha \in (1,2)$ using a probabilistic argument in Theorem 4.3, which is similar to [2]. We would like to mention that in Theorem 4.3, we express the 2nd coefficient of $|D| - \tilde{Q}_D^{(\alpha)}(t)$ by means of the probabilistic term $\mathbb{E}\left[\overline{W}_{S_1^{(\alpha/2)}}\right]$, which is more natural than other previously known expressions (compare it with [11, Theorem 1.1]). In order to prove the second part of Theorem 1.2 ($\alpha = 1$), we establish the tail probability $\mathbb{P}\left(\overline{W}_{S_t^{(\alpha/2)}} > u\right)$ for $u > 1$ in Proposition 4.7, which is an amusingly simple expression. Once having established Proposition 4.7, it is straightforward to compute the difference $\mathbb{P}\left(\overline{W}_{S_1^{(\alpha/2)}} > u\right) - \frac{2}{\pi u}$ for large u. Then, we prove that main terms of order $t \ln(1/t)$ cancel out each other again, and show that the remaining terms are of order t.

The organization of this paper is as follows. In Section 2, we introduce notations and recall some preliminary facts. In Section 3, we study the spectral heat content for killed subordinate Brownian motions and prove Theorem 1.1. The first part of Theorem 1.1 is proved in the Section 3.1, and the second part of Theorem 1.1 is proved in the Section 3.2. In Section 4, we study the spectral heat content for subordinate killed Brownian motions, and prove first and second parts of Theorem 1.2 in Sections 4.1 and 4.2, respectively. The notation $\mathbb{P}_x$ stands for the law of the underlying processes started at $x \in \mathbb{R}$, and $\mathbb{E}_x$ stands for expectation with respect to $\mathbb{P}_x$. For simplicity, we use $\mathbb{P} = \mathbb{P}_0$ and $\mathbb{E} = \mathbb{E}_0$.

2 Preliminaries

In this section, we introduce some notations and define the functions to be studied in the later sections. All stochastic processes and domains are one dimensional objects.

Let $W = \{W_t\}_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$. The density of the gaussian random variable W_t is $p(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ with the characteristic function given by

$$\mathbb{E}\left[e^{i\xi W_t}\right] = e^{-t\xi^2}, \quad \xi \in \mathbb{R}.$$

The supremum process $\overline{W} = \left\{\overline{W}_t\right\}_{t\geq 0}$ of the Brownian motion is defined by $\overline{W}_t = \sup_{s\leq t} W_s$. It follows from [10, Theorem 2.21] that $|W_t|$ and $\overline{W}_t$ have the same distribution.

Let $S^{(\alpha/2)} = \left\{S_t^{(\alpha/2)}\right\}_{t\geq 0}$ be an $\alpha/2$-stable subordinator. That is, $S^{(\alpha/2)}$ is an increasing Lévy process started at zero whose Laplace exponent is

$$\mathbb{E}\left[e^{-\lambda S_t^{(\alpha/2)}}\right] = e^{-t\lambda^{\alpha/2}}, \quad \lambda \geq 0. \tag{2.1}$$

It follows from Eq. 2.1 that $S_t^{(\alpha/2)}$ and $t^{2/\alpha} S_1^{(\alpha/2)}$ have the same distribution for any $t > 0$. The subordinator $S^{(\alpha/2)}$ is an increasing process started at 0, and for this reason it plays a role as time. By doing an elementary integral, it is easy to check that

$$\lambda^{\alpha/2} = \frac{\alpha/2}{\Gamma\left(1-\frac{\alpha}{2}\right)} \int_0^\infty \left(1 - e^{-\lambda t}\right) t^{-1-\frac{\alpha}{2}}\, dt, \quad \lambda > 0, \alpha \in (0,2).$$

This shows that the Lévy density $j^{SS}(u)$ for $S^{(\alpha/2)}$ is

$$j^{SS}(u) = \frac{\alpha/2}{\Gamma\left(1-\frac{\alpha}{2}\right)} u^{-1-\frac{\alpha}{2}}, \quad u > 0. \tag{2.2}$$

It follows from [11, Equation (2.3)] or [7, Equation (18)] that the density $g_1^{(\alpha/2)}(x)$ of $S_1^{(\alpha/2)}$ exists, and is given by

$$g_1^{(\alpha/2)}(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma\left(1+\frac{\alpha n}{2}\right)}{n!} \sin\left(\frac{\pi\alpha n}{2}\right) x^{-\frac{\alpha n}{2}-1}, \quad x > 0. \tag{2.3}$$

It follows from the scaling property (2.1) that we have

$$g_t^{(\alpha/2)}(x) = t^{-2/\alpha} g_1^{(\alpha/2)}\left(\frac{x}{t^{2/\alpha}}\right). \tag{2.4}$$

Now we define subordinate Brownian motions. Let W and $S^{(\alpha/2)}$ be Brownian motions and stable subordinators defined on some probability space. Assume that they are independent. Then, the subordinate Brownian motions by the subordinator $S^{(\alpha/2)}$ are the following time-changed Brownian motions:

$$X_t^{(\alpha)} := W_{S_t^{(\alpha/2)}}.$$

By conditioning on $S_t^{(\alpha/2)}$, one can observe that the characteristic function of time changed process $X^{(\alpha)} = \left\{X_t^{(\alpha)}\right\}_{t\geq 0} := \left\{W_{S_t^{(\alpha/2)}}\right\}_{t\geq 0}$ is given by

$$\mathbb{E}\left[e^{i\xi X_t^{(\alpha)}}\right] = \mathbb{E}\left[e^{i\xi W_{S_t^{(\alpha/2)}}}\right] = \mathbb{E}\left[e^{-S_t^{(\alpha/2)}\xi^2}\right] = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}, \tag{2.5}$$

and this shows that $X^{(\alpha)}$ are symmetric stable processes of index α. From Eq. 2.5, we observe that $X_t^{(\alpha)}$ has the scaling property; $X_t^{(\alpha)}$ and $t^{1/\alpha}X_1^{(\alpha)}$ have the same distribution for any $t > 0$. The Lévy density $j^{SSP}(x)$ of $X^{(\alpha)}$ is given by (see [6, Equation (1.3) and (1.22)])

$$j^{SSP}(x) = \frac{A_{1,\alpha}}{|x|^{1+\alpha}}, \quad A_{1,\alpha} = \frac{\alpha 2^{\alpha-1}\Gamma\left(\frac{1+\alpha}{2}\right)}{\pi^{1/2}\Gamma\left(1-\frac{\alpha}{2}\right)}. \tag{2.6}$$

Let D be an open set in $\mathbb{R}$, and define $\tau_D^{(\alpha)} = \inf\{t > 0 : X_t^{(\alpha)} \notin D\}$ be the first exit time from D by $X^{(\alpha)}$. The killed processes $X^{(\alpha),D} = \left\{X_t^{(\alpha),D}\right\}_{t\geq 0}$ are defined by

$$X_t^{(\alpha),D} = \begin{cases} X_t^{(\alpha)} & \text{if } t < \tau_D^{(\alpha)}, \\ \partial & \text{if } t \geq \tau_D^{(\alpha)}, \end{cases}$$

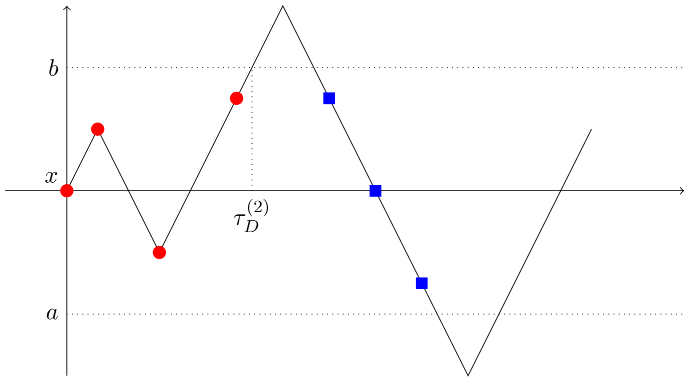
where ∂ is a cemetery state. The process $X^{(\alpha),D}$ will be called *killed subordinate Brownian motions* (by stable subordinators $S^{(\alpha/2)}$), since we first subordinate (time-change) Brownian motions, then kill the process when they exit the domain. We can exchange the order of time-change and killing, and the corresponding process will be called *subordinate killed Brownian motions* ((by stable subordinators $S^{(\alpha/2)}$). More precisely, let $\tau_D^{(2)} = \inf\{t > 0 : W_t \notin D\}$ be the first exit time from D by Brownian motions W. Define killed Brownian motions $W^D = \{W_t^D\}_{t\geq 0}$ as

$$W_t^D = \begin{cases} W_t & \text{if } t < \tau_D^{(2)}, \\ \partial & \text{if } t \geq \tau_D^{(2)}. \end{cases}$$

Now the subordinate killed Brownian motions $\left(W^D\right)_{S^{(\alpha/2)}} = \left\{\left(W^D\right)_{S_t^{(\alpha/2)}}\right\}_{t\geq 0}$ are defined by

$$\left(W^D\right)_{S_t^{(\alpha/2)}} = \begin{cases} W_{S_t^{(\alpha/2)}} & \text{if } S_t^{(\alpha/2)} < \tau_D^{(2)}, \\ \partial & \text{if } S_t^{(\alpha/2)} \geq \tau_D^{(2)}. \end{cases}$$

The following graph illustrates sample paths of $X^{(\alpha),D}$ and $\left(W^D\right)_{S^{(\alpha/2)}}$ starting from x when $D = (a, b)$, where the straight lines represent the sample paths of Brownian motions, the circles represent the sample paths of $\left(W^D\right)_{S^{(\alpha/2)}}$, while the circles together with the rectangles represent the sample paths of $X^{(\alpha),D}$.



Sample paths of $\left(W^D\right)_{S^{(\alpha/2)}}$ and $X^{(\alpha),D}$

Let $\zeta = \inf\{r \geq 0 : \left(W^D\right)_{S_r^{(\alpha/2)}} = \partial\}$ be the life time of $\left(W^D\right)_{S_t^{(\alpha/2)}}$. Then, we have

$$\{\zeta > t\} = \left\{\tau_D^{(2)} > S_t^{(\alpha/2)}\right\}.$$

Clearly, we have $\{\zeta > t\} \subset \left\{\tau_D^{(\alpha)} > t\right\}$, and the inclusion can be strict.

We define the supremum processes $\overline{X}^{(\alpha)} = \left\{\overline{X}_t^{(\alpha)}\right\}_{t\geq 0}$ of the stable processes as

$$\overline{X}_t^{(\alpha)} := \sup_{u\leq t} X_u^{(\alpha)} = \sup_{u\leq t} W_{S_u^{(\alpha/2)}}. \tag{2.7}$$

Similarly, $\overline{W}_{S^{(\alpha/2)}} = \left\{\overline{W}_{S_t^{(\alpha/2)}}\right\}_{t\geq 0}$ are defined by

$$\overline{W}_{S_t^{(\alpha/2)}} = \sup_{u\leq S_t^{(\alpha/2)}} W_u. \tag{2.8}$$

It is noteworthy to mention that even though two expressions $X^{(\alpha)}$ and $W_{S^{(\alpha/2)}}$ mean the same objects, stable processes of index α, the supremum notations $\overline{X}^{(\alpha)}$ and $\overline{W}_{S^{(\alpha/2)}}$ are *different*, and we always have $\overline{X}_t^{(\alpha)} \leq \overline{W}_{S_t^{(\alpha/2)}}$. The infimum processes $\underline{W}$, $\underline{X}^{(\alpha)}$, and $\underline{W}_{S^{(\alpha/2)}}$ are defined in similar ways with the supremum being replaced by the infimum.

Finally, we define the spectral heat content $Q_D^{(\alpha)}(t)$ and $\tilde{Q}_D^{(\alpha)}(t)$ for killed subordinate Brownian motions and subordinate killed Brownian motions. The spectral heat content $Q_D^{(\alpha)}(t)$ for killed subordinate Brownian motions is defined by

$$Q_D^{(\alpha)}(t) := \int_D \mathbb{P}_x(\tau_D^{(\alpha)} > t)dx,$$

and the spectral heat content $\tilde{Q}_D^{(\alpha)}(t)$ for subordinate killed Brownian motions is defined by

$$\tilde{Q}_D^{(\alpha)}(t) := \int_D \mathbb{P}_x(\zeta > t)dx = \int_D \mathbb{P}_x\left(\tau_D^{(2)} > S_t^{(\alpha/2)}\right)dx.$$

Since $\{\zeta > t\} \subset \left\{\tau_D^{(\alpha)} > t\right\}$, we always have

$$\tilde{Q}_D^{(\alpha)}(t) \le Q_D^{(\alpha)}(t).$$

When $X^{(\alpha)}$ starts at $x \in (a, b)$, we have

$$\left\{\tau_D^{(\alpha)} \le t\right\} = \left\{\overline{X}_t^{(\alpha)} \ge b \text{ or } \underline{X}_t^{(\alpha)} \le a\right\}.$$

It follows from the scaling property and the symmetry of $X^{(\alpha)}$, an elementary probability law $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ for any events A and B, and the change of variable $u = (b - x)t^{-1/\alpha}$ and $v = (a - x)t^{-1/\alpha}$, we have

$$\begin{aligned}
|D| - Q_D^{(\alpha)}(t) &= \int_D \mathbb{P}_x(\tau_D^{(\alpha)} \le t)dx = \int_a^b \mathbb{P}_x\left(\overline{X}_t^{(\alpha)} \ge b \text{ or } \underline{X}_t^{(\alpha)} \le a\right)dx \\
&= \int_a^b \mathbb{P}\left(\overline{X}_t^{(\alpha)} \ge b - x\right)dx + \int_a^b \mathbb{P}\left(\underline{X}_t^{(\alpha)} \le a - x\right)dx \\
&\quad - \int_a^b \mathbb{P}_x\left(\overline{X}_t^{(\alpha)} \ge b \text{ and } \underline{X}_t^{(\alpha)} \le a\right)dx \\
&= \int_a^b \mathbb{P}\left(\overline{X}_1^{(\alpha)} \ge (b - x)t^{-1/\alpha}\right)dx + \int_a^b \mathbb{P}\left(\underline{X}_1^{(\alpha)} \le (a - x)t^{-1/\alpha}\right)dx \\
&\quad - \int_a^b \mathbb{P}_x\left(\overline{X}_t^{(\alpha)} \ge b \text{ and } \underline{X}_t^{(\alpha)} \le a\right)dx \\
&= t^{1/\alpha}\int_0^{\frac{b-a}{t^{1/\alpha}}} \mathbb{P}\left(\overline{X}_1 \ge u\right)du + t^{1/\alpha}\int_{-\frac{b-a}{t^{1/\alpha}}}^0 \mathbb{P}\left(\underline{X}_1^{(\alpha)} \le v\right)dv \\
&\quad - \int_a^b \mathbb{P}_x\left(\overline{X}_t^{(\alpha)} \ge b \text{ and } \underline{X}_t^{(\alpha)} \le a\right)dx \\
&= 2t^{1/\alpha}\int_0^{\frac{b-a}{t^{1/\alpha}}} \mathbb{P}\left(\overline{X}_1 \ge u\right)du - \int_a^b \mathbb{P}_x\left(\overline{X}_t^{(\alpha)} \ge b \text{ and } \underline{X}_t^{(\alpha)} \le a\right)dx. \qquad (2.9)
\end{aligned}$$

3 Spectral Heat Content for Killed Subordinate Brownian Motions

3.1 Case: $\alpha \in (1, 2)$

We start with a simple lemma. Let $p_t^{(\alpha)}(x)$ be the transition density (heat kernel) for $X_t^{(\alpha)}$. Note that the following heat kernel estimate is well-known (see [5]);

$$c^{-1}\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right) \le p_t^{(\alpha)}(x) \le c\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right) \qquad (3.1)$$

for some constant $c > 1$.

Lemma 3.1 *Suppose that* $1 < \alpha < 2$. *Then*

$$\lim_{t\to 0} \frac{t^{1/\alpha}\int_{(b-a)t^{-1/\alpha}}^{\infty} \mathbb{P}\left(\overline{X}_1^{(\alpha)} > u\right)du}{t} = \frac{2^{\alpha-1}\Gamma\left(\frac{1+\alpha}{2}\right)}{(\alpha - 1)\pi^{1/2}\Gamma\left(1 - \frac{\alpha}{2}\right)(b - a)^{\alpha-1}}.$$

Proof It follows from L'Hôpital's rule, the scaling property of $X_t^{(\alpha)}$, [3, Proposition VIII 4], [12, Corollary 8.9], and (2.6) that we have

$$\lim_{t\to 0}\frac{\int_{(b-a)t^{-1/\alpha}}^{\infty}\mathbb{P}\left(\overline{X}_1^{(\alpha)}>u\right)du}{t^{1-1/\alpha}}=\lim_{t\to 0}\frac{(b-a)}{\alpha-1}\frac{\mathbb{P}\left(\overline{X}_1^{(\alpha)}>(b-a)t^{-1/\alpha}\right)}{t}$$

$$=\lim_{t\to 0}\frac{(b-a)}{\alpha-1}\frac{\mathbb{P}\left(X_1^{(\alpha)}>(b-a)t^{-1/\alpha}\right)}{t}=\lim_{t\to 0}\frac{(b-a)}{\alpha-1}\frac{\mathbb{P}\left(X_t^{(\alpha)}>b-a\right)}{t}$$

$$=\lim_{t\to 0}\frac{(b-a)}{\alpha-1}\int_{b-a}^{\infty}\frac{p_t^{(\alpha)}(u)}{t}du=\frac{(b-a)}{\alpha-1}\int_{b-a}^{\infty}\frac{A_{1,\alpha}}{u^{1+\alpha}}du$$

$$=\frac{2^{\alpha-1}\Gamma\left(\frac{1+\alpha}{2}\right)}{(\alpha-1)\pi^{1/2}\Gamma\left(1-\frac{\alpha}{2}\right)(b-a)^{\alpha-1}}.$$

□

Lemma 3.2 *Let* $\alpha\in(1,2)$. *Then, for any* $t>0$, *we have*

$$\int_a^b\mathbb{P}_x\left(\overline{X}_t^{(\alpha)}>b\ \text{and}\ \underline{X}_t^{(\alpha)}<a\right)dx\le\frac{ct^{1+\frac{1}{\alpha}}}{(b-a)^{\alpha}}\mathbb{E}\left[\overline{X}_1^{(\alpha)}\right]$$

for some constant $c>0$.

Proof Define

$$\tau:=\inf\left\{u: X_u^{(\alpha)}>b\ \text{or}\ X_u^{(\alpha)}<a\right\}.$$

Clearly, τ is a stopping time with respect to the natural filtration $\mathcal{F}=\{\mathcal{F}_t\}_{t\ge 0}$. When the process $X^{(\alpha)}$ starts at $x\in(a,b)$, we have

$$\begin{aligned}
&\left\{\overline{X}_t^{(\alpha)}>b\ \text{and}\ \underline{X}_t^{(\alpha)}<a\right\}=\left\{\tau<t,\overline{X}_t^{(\alpha)}>b\ \text{and}\ \underline{X}_t^{(\alpha)}<a\right\}\\
&=\left\{\tau<t,X_\tau^{(\alpha)}\le a,\overline{X}_t^{(\alpha)}>b\ \text{and}\ \underline{X}_t^{(\alpha)}<a\right\}\cup\left\{\tau<t,X_\tau^{(\alpha)}\ge b,\overline{X}_t^{(\alpha)}>b\ \text{and}\ \underline{X}_t^{(\alpha)}<a\right\}\\
&\subset\left\{\tau<t,X_\tau^{(\alpha)}\le a,\overline{X}_t^{(\alpha)}>b\right\}\cup\left\{\tau<t,X_\tau^{(\alpha)}\ge b,\underline{X}_t^{(\alpha)}<a\right\}\\
&\subset\left\{\tau<t,\sup_{\tau\le s\le t}\left(X_s^{(\alpha)}-X_\tau^{(\alpha)}\right)>b-a\right\}\cup\left\{\tau<t,\inf_{\tau\le s\le t}\left(X_s^{(\alpha)}-X_\tau^{(\alpha)}\right)<-(b-a)\right\}\\
&\subset\left\{\tau<t,\sup_{0\le s\le t}\left(X_{s+\tau}^{(\alpha)}-X_\tau^{(\alpha)}\right)>b-a\right\}\cup\left\{\tau<t,\inf_{0\le s\le t}\left(X_{s+\tau}^{(\alpha)}-X_\tau^{(\alpha)}\right)<-(b-a)\right\}\\
&=\left\{\tau<t,\overline{Y}_t>b-a\right\}\cup\left\{\tau<t,\underline{Y}_t<-(b-a)\right\},
\end{aligned}$$

where $Y_u:=X_{u+\tau}^{(\alpha)}-X_\tau^{(\alpha)}$. By the strong Markov property, Y has the same distribution as $X^{(\alpha)}$ started from 0, and is independent of $\mathcal{F}_\tau$. Hence, for any $x\in(a,b)$, we have

$$\begin{aligned}
&\mathbb{P}_x\left(\overline{X}_t^{(\alpha)}>b\ \text{and}\ \underline{X}_t^{(\alpha)}<a\right)\\
&\le\mathbb{P}_x\left(\tau<t,\overline{Y}_t>b-a\right)+\mathbb{P}_x\left(\tau<t,\underline{Y}_t<-(b-a)\right)\\
&=\mathbb{P}_x(\tau<t)\mathbb{P}\left(\overline{Y}_t>b-a\right)+\mathbb{P}_x(\tau<t)\mathbb{P}(\underline{Y}_t<-(b-a))\\
&=\mathbb{P}_x(\tau<t)\mathbb{P}\left(\overline{X}_t^{(\alpha)}>b-a\right)+\mathbb{P}_x(\tau<t)\mathbb{P}\left(\underline{X}_t^{(\alpha)}<-(b-a)\right)\\
&=2\mathbb{P}_x(\tau<t)\mathbb{P}\left(\overline{X}_t^{(\alpha)}>b-a\right),
\end{aligned}\tag{3.2}$$

where we used the fact that $\overline{X}_t^{(\alpha)}$ and $-\underline{X}_t^{(\alpha)}$ have the same distribution because of the symmetry of $X^{(\alpha)}$.

From the scaling property of $X^{(\alpha)}$, (3.1), and [2, Proposition 2.1], we have

$$\mathbb{P}\left(\overline{X}_t^{(\alpha)} > b-a\right) = \mathbb{P}\left(\overline{X}_1^{(\alpha)} > \frac{b-a}{t^{1/\alpha}}\right) \leq 2\mathbb{P}\left(X_1^{(\alpha)} > \frac{b-a}{t^{1/\alpha}}\right) \leq c_1 \int_{(b-a)t^{-1/\alpha}}^{\infty} \frac{1}{u^{1+\alpha}}du \leq \frac{c_2 t}{(b-a)^{\alpha}}. \quad (3.3)$$

When X starts at $x \in (a, b)$, we have

$$\{\tau < t\} = \left\{\overline{X}_t^{(\alpha)} > b \text{ or } \underline{X}_t^{(\alpha)} < a\right\}.$$

Hence, from Eqs. 3.2 and (3.3), we have

$$\int_a^b \mathbb{P}_x\left(\overline{X}_t^{(\alpha)} > b \text{ and } \underline{X}_t^{(\alpha)} < a\right)dx \leq \frac{c_3 t}{(b-a)^{\alpha}} \int_a^b \mathbb{P}_x\left(\overline{X}_t^{(\alpha)} > b \text{ or } \underline{X}_t^{(\alpha)} < a\right)dx$$
$$\leq \frac{c_3 t}{(b-a)^{\alpha}} \left(\int_a^b \mathbb{P}_x\left(\overline{X}_t^{(\alpha)} > b\right)dx + \int_a^b \mathbb{P}_x\left(\underline{X}_t^{(\alpha)} < a\right)dx\right).$$

By the scaling property of X and the change of variable $u = (b-x)t^{-1/\alpha}$, we have

$$\int_a^b \mathbb{P}_x\left(\overline{X}_t^{(\alpha)} > b\right)dx = \int_a^b \mathbb{P}\left(\overline{X}_1^{(\alpha)} > (b-x)t^{-1/\alpha}\right) = t^{1/\alpha}\int_0^{(b-a)t^{-1/\alpha}} \mathbb{P}\left(\overline{X}_1^{(\alpha)} > u\right)du \leq t^{1/\alpha}\mathbb{E}\left[\overline{X}_1^{(\alpha)}\right].$$

Similarly, by the change of variable $v = (x-a)t^{-1/\alpha}$, and the fact that $\overline{X}_t$ and $-\underline{X}_t$ have the same distribution, we have

$$\int_a^b \mathbb{P}_x\left(\underline{X}_t^{(\alpha)} < a\right)dx = \int_a^b \mathbb{P}\left(\overline{X}_1^{(\alpha)} > (x-a)t^{-1/\alpha}\right)$$
$$= t^{1/\alpha}\int_0^{(b-a)t^{-1/\alpha}} \mathbb{P}\left(\overline{X}_1^{(\alpha)} > u\right)du \leq t^{1/\alpha}\mathbb{E}\left[\overline{X}_1^{(\alpha)}\right].$$

Now the conclusion follows immediately. □

Now we are ready to prove the first part of Theorem 1.1.

Proof of (1.1)

From Eq. 2.9, we have

$$|D| - Q_D^{(\alpha)}(t) - \mathbb{E}\left[\overline{X}_1^{(\alpha)}\right]|\partial D|t^{1/\alpha} = \int_a^b \mathbb{P}\left(\tau_D^{(\alpha)} \leq t\right)dx - \mathbb{E}\left[\overline{X}_1^{(\alpha)}\right]|\partial D|t^{1/\alpha}$$
$$= 2t^{1/\alpha}\int_0^{\frac{b-a}{t^{1/\alpha}}} \mathbb{P}\left(\overline{X}_1^{(\alpha)} > u\right)du - \int_a^b \mathbb{P}_x\left(\overline{X}_t^{(\alpha)} > b \text{ and } \underline{X}_t^{(\alpha)} < a\right)dx$$
$$-2t^{1/\alpha}\int_0^{\infty} \mathbb{P}\left(\overline{X}_1^{(\alpha)} > u\right)du$$
$$= 2t^{1/\alpha}\int_{\frac{b-a}{t^{1/\alpha}}}^{\infty} \mathbb{P}\left(\overline{X}_1^{(\alpha)} > u\right)du - \int_a^b \mathbb{P}_x\left(\overline{X}_t^{(\alpha)} > b \text{ and } \underline{X}_t^{(\alpha)} < a\right)dx. \quad (3.4)$$

Now the conclusion follows immediately from Lemmas 3.1 and 3.2. □

3.2 Case: $\alpha = 1$

In this subsection, we study the asymptotic behavior of the spectral heat content for killed subordinate Brownian motions (killed stable processes) when $\alpha = 1$. We start with a lemma that is similar to Lemma 3.2.

Lemma 3.3

$$\int_a^b \mathbb{P}_x\left(\overline{X}_t^{(1)} > b \text{ and } \underline{X}_t^{(1)} < a\right) dx = O\left(t^2 \ln(1/t)\right) \text{ as } t \to 0.$$

Proof The proof is similar to the proof of Lemma 3.2, and we only explain the difference. As in the proof of Lemma 3.2, we have

$$\int_a^b \mathbb{P}_x\left(\tau < t, \sup_{\tau \le s \le t} X_s^{(1)} - X_\tau^{(1)} > b - a\right) dx \le \frac{ct}{(b-a)^\alpha} \int_a^b \mathbb{P}_x\left(\overline{X}_t^{(1)} > b\right) dx$$
$$\le \frac{ct^2}{(b-a)^\alpha} \int_0^{\frac{b-a}{t^{1/\alpha}}} \mathbb{P}\left(\overline{X}_1^{(1)} > u\right) du = O\left(t^2 \ln(1/t)\right),$$

where the last part comes from [2, Proposition 4.3.(i)]. □

There was an error in the paragraph right above [2, Remark 5.1]. The density for $\overline{X}_1^{(1)}$ exists and it is given by (see [8])

$$f(x) = \frac{1}{\pi x^{1/2}\left(1+x^2\right)^{3/4}} \exp\left(-\frac{1}{\pi}\int_0^{1/x} \frac{\ln v}{1+v^2} dv\right), \quad x > 0. \tag{3.5}$$

We note that there is also a minor error in the exact expression of $f(x)$ in [8] and the upper bound of the integral should be $\frac{1}{x}$, instead of x.

Now we are ready to prove the second part of Theorem 1.1.

Proof of (1.2)

From Eq. 2.9, we have

$$|D| - Q_D^{(1)}(t) = \int_a^b \mathbb{P}\left(\tau_D^{(1)} \le t\right) dx = 2t \int_0^{\frac{b-a}{t}} \mathbb{P}\left(\overline{X}_1^{(1)} > u\right) du$$
$$- \int_a^b \mathbb{P}_x\left(\overline{X}_t^{(1)} > b \text{ and } \underline{X}_t^{(1)} < a\right) dx.$$

It follows from Lemma 3.3.

$$\lim_{t\to 0} \frac{\int_a^b \mathbb{P}_x\left(\overline{X}_t^{(1)} > b \text{ and } \underline{X}_t^{(1)} < a\right) dx}{t} = 0.$$

Note that from [2, Proposition 4.3.(i)], we have

$$\lim_{t\to 0} \frac{2t\int_0^{\frac{b-a}{t}} \mathbb{P}\left(\overline{X}_1^{(1)} > u\right) du}{t\ln(1/t)} = \frac{2}{\pi}.$$

We will show that

$$\lim_{t\to 0} \frac{t\int_0^{\frac{b-a}{t}} \mathbb{P}\left(\overline{X}_1^{(1)} > u\right) du - \frac{t\ln(1/t)}{\pi}}{t} = \lim_{t\to 0}\left(\int_0^{\frac{b-a}{t}} \mathbb{P}\left(\overline{X}_1^{(1)} > u\right) du - \frac{\ln(1/t)}{\pi}\right)$$
$$= \int_0^1 \mathbb{P}\left(\overline{X}_1^{(1)} > u\right) du + \frac{\ln(b-a)}{\pi} + \int_1^\infty \left(\mathbb{P}\left(\overline{X}_1^{(1)} > u\right) - \frac{1}{\pi u}\right) du.$$

Note for any $0 < t < b-a$ that

$$
\begin{aligned}
&\int_0^{\frac{b-a}{t}} \mathbb{P}\left(\overline{X}_1^{(1)} > u\right) du - \frac{\ln(1/t)}{\pi} \\
&= \int_0^1 \mathbb{P}\left(\overline{X}_1^{(1)} > u\right) du + \int_1^{\frac{b-a}{t}} \mathbb{P}\left(\overline{X}_1^{(1)} > u\right) du - \int_1^{\frac{b-a}{t}} \frac{1}{\pi u} du + \frac{\ln(b-a)}{\pi} \\
&= \int_0^1 \mathbb{P}\left(\overline{X}_1^{(1)} > u\right) du + \frac{\ln(b-a)}{\pi} + \int_1^{\frac{b-a}{t}} \left(\mathbb{P}\left(\overline{X}_1^{(1)} > u\right) - \frac{1}{\pi u}\right) du.
\end{aligned}
$$

It follows from Eq. 3.5 and the change of variable $y = \frac{1}{v}$, we have

$$
\mathbb{P}\left(\overline{X}_1^{(1)} > u\right) = \int_u^\infty \frac{1}{\pi x^{1/2} \left(1+x^2\right)^{3/4}} \exp\left(\frac{1}{\pi} \int_x^\infty \frac{\ln y}{1+y^2} dy\right) dx.
$$

We will show that for all sufficiently large u, we have

$$
\left|\mathbb{P}\left(\overline{X}_1^{(1)} > u\right) - \frac{1}{\pi u}\right| \le \frac{4}{\pi^2} \frac{\ln u}{u^2}, \tag{3.6}
$$

so that by the Lebesgue dominated convergence theorem,

$$
\lim_{t\to 0} \int_1^{\frac{b-a}{t}} \left(\mathbb{P}\left(\overline{X}_1^{(1)} > u\right) - \frac{1}{\pi u}\right) du = \int_1^\infty \left(\mathbb{P}\left(\overline{X}_1^{(1)} > u\right) - \frac{1}{\pi u}\right) du.
$$

For $u \ge 1$ and $x \ge u$, we have $\exp\left(\frac{1}{\pi}\int_x^\infty \frac{\ln y}{1+y^2} dy\right) \ge e^0 = 1$ and

$$
\begin{aligned}
&\int_u^\infty \frac{1}{\pi x^{1/2} \left(1+x^2\right)^{3/4}} \exp\left(\frac{1}{\pi} \int_x^\infty \frac{\ln y}{1+y^2} dy\right) dx \ge \int_u^\infty \frac{1}{\pi x^{1/2} \left(1+x^2\right)^{3/4}} dx \\
&\ge \int_u^\infty \frac{1}{\pi \left(1+x^2\right)} dx \\
&= \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan u\right) = \frac{1}{\pi}\arctan(1/u) = \sum_{n=0}^\infty \frac{(-1)^n}{\pi(2n+1)} \frac{1}{u^{2n+1}},
\end{aligned}
$$

where we used an elementary identity $\arctan u + \arctan(1/u) = \frac{\pi}{2}$. Hence, there exists $U_1 > 0$ such that

$$
\mathbb{P}\left(\overline{X}_1^{(1)} > u\right) - \frac{1}{\pi u} \ge -\frac{1}{2\pi} \frac{1}{u^3}, \text{ for all } u \ge U_1. \tag{3.7}
$$

Now we focus on establishing the upper bound. From Karamata's Theorem ([4, Theorem 1.5.11 (ii)]), we have

$$
\int_x^\infty \frac{\ln y}{1+y^2} dy = \int_x^\infty y^{-2} \frac{y^2 \ln y}{1+y^2} dy \sim \frac{x \ln x}{1+x^2} \text{ as } x \to \infty.
$$

Hence, there exists $U_2 > 0$ such that for all $x \ge u \ge U_2$, we have

$$
\int_x^\infty \frac{\ln y}{1+y^2} dy \le \frac{2x \ln x}{1+x^2}. \tag{3.8}
$$

By an elementary calculus, we see that $e^u \le 1 + 2u$ for all $0 \le u \le \ln 2$, and take U_3 so that

$$
\frac{2}{\pi} \frac{x \ln x}{1+x^2} \le \ln 2 \text{ for all } x \ge u \ge U_3. \tag{3.9}
$$

It follows from Eqs. 3.8 and 3.9 for $u \geq \max(U_2, U_3)$, we have

$$
\begin{aligned}
&\mathbb{P}\left(\overline{X}_1^{(1)} > u\right) - \frac{1}{\pi u} \\
&\leq \int_u^\infty \frac{1}{\pi x^{1/2}\left(1+x^2\right)^{3/4}} \exp\left(\frac{2}{\pi}\frac{x\ln x}{1+x^2}\right) dx - \frac{1}{\pi u} \\
&\leq \int_u^\infty \frac{1}{\pi x^{1/2}\left(1+x^2\right)^{3/4}} \left(1+\frac{4}{\pi}\frac{x\ln x}{1+x^2}\right) dx - \frac{1}{\pi u} \\
&\leq \int_u^\infty \frac{1}{\pi x^2} dx + \int_u^\infty \frac{1}{\pi x^{1/2}\left(1+x^2\right)^{3/4}} \frac{4}{\pi}\frac{x\ln x}{1+x^2} dx - \frac{1}{\pi u} \\
&= \frac{4}{\pi^2}\int_u^\infty \frac{x^{1/2}\ln x}{\left(1+x^2\right)^{7/4}} dx \leq \frac{4}{\pi^2}\int_u^\infty \frac{\ln x}{x^3} dx.
\end{aligned}
$$

Again, it follows from [4, Theorem 1.5.11 (ii)], we have

$$\int_u^\infty \frac{\ln x}{x^3} dx \sim \frac{1}{2}\frac{\ln u}{u^2} \text{ as } u \to \infty,$$

and we can take a constant $U_4 \geq \max(U_2, U_3)$ such that $\int_u^\infty \frac{\ln x}{x^3} dx \leq \frac{\ln u}{u^2}$ for all $u \geq U_4$. Hence, for $u \geq U_4$

$$\mathbb{P}\left(\overline{X}_1^{(1)} > u\right) - \frac{1}{\pi u} \leq \frac{4}{\pi^2}\frac{\ln u}{u^2}. \tag{3.10}$$

Hence, it follows from Eqs. 3.7 and 3.10, there exists $U_5 \geq \max(U_1, U_4)$ such that (3.6) holds for all $u \geq U_5$. □

4 Spectral Heat Content for Subordinate Killed Brownian Motions

In this section, we study the 3rd term of the spectral heat content for subordinate killed Brownian motions, and prove Theorem 1.2.

4.1 Case: $\alpha \in (1, 2)$

Lemma 4.1 *For any $\alpha \in (0, 2)$, there exists a constant $c = c(\alpha) > 0$ such that*

$$\mathbb{P}\left(\overline{W}_{S_t^{(\alpha/2)}} > b-a\right) \leq ct \textit{ for all } t > 0.$$

Proof By the scaling property and [10, Theorem 2.21], we have

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{u \leq S_t^{(\alpha/2)}} W_u > b-a\right) = \mathbb{P}\left(|W_{S_t^{(\alpha/2)}}| > b-a\right) = \mathbb{P}\left(\left(S_t^{(\alpha/2)}\right)^{1/2}|W_1| > b-a\right) \\
&= \mathbb{P}\left(t^{1/\alpha}\left(S_1^{(\alpha/2)}\right)^{1/2} > \frac{b-a}{|W_1|}\right) = \mathbb{P}\left(S_1^{(\alpha/2)} > \frac{(b-a)^2}{t^{2/\alpha}|W_1|^2}\right).
\end{aligned}
$$

Hence, we have

$$\mathbb{P}\left(\sup_{u\leq S_t^{(\alpha/2)}} W_u > b-a\right) = 2\int_0^\infty \mathbb{P}\left(S_1^{(\alpha/2)} > \frac{(b-a)^2}{t^{2/\alpha}x^2}\right)\frac{1}{\sqrt{4\pi}}e^{-\frac{x^2}{4}}dx$$

$$= 2\int_0^\infty \left(\frac{(b-a)^2}{t^{2/\alpha}x^2}\right)^{\alpha/2} \mathbb{P}\left(S_1^{(\alpha/2)} > \frac{(b-a)^2}{t^{2/\alpha}x^2}\right)\frac{1}{\sqrt{4\pi}}e^{-\frac{x^2}{4}}$$
$$\left(\frac{(b-a)^2}{t^{2/\alpha}x^2}\right)^{-\alpha/2} dx$$

$$= \frac{t}{\sqrt{\pi}(b-a)^\alpha}\int_0^\infty \left(\frac{(b-a)^2}{t^{2/\alpha}x^2}\right)^{\alpha/2} \mathbb{P}\left(S_1^{(\alpha/2)} > \frac{(b-a)^2}{t^{2/\alpha}x^2}\right)$$
$$\times x^\alpha e^{-\frac{x^2}{4}}dx.$$

It follows from [11, Equation (2.8)], there exists a constant c_1 such that for all $u \in (0,\infty)$,

$$u^{\alpha/2}\mathbb{P}\left(S_1^{(\alpha/2)} > u\right) \leq c_1.$$

Hence, we have

$$\mathbb{P}\left(\sup_{u\leq S_t} W_u > b-a\right) \leq \frac{c_1 t}{\sqrt{\pi}(b-a)^\alpha}\int_0^\infty x^\alpha e^{-\frac{x^2}{4}}dx.$$

□

Lemma 4.2 *Let* $\alpha \in (1,2)$. *Then, there exists a constant* $c = c(\alpha) > 0$ *such that*

$$\int_a^b \mathbb{P}_x\left(\sup_{u\leq S_t^{(\alpha/2)}} W_u > b \textit{ and } \inf_{u\leq S_t^{(\alpha/2)}} W_u < a\right)dx \leq c\mathbb{E}\left[\overline{W}_{S_1^{(\alpha/2)}}\right]t^{1+\frac{1}{\alpha}}.$$

Proof The proof is similar to the proof of Lemma 3.2, and we provide the details for the reader's convenience. Define

$$\eta := \inf\left\{v : \sup_{u\leq S_v^{(\alpha/2)}} W_u > b \text{ or } \inf_{u\leq S_v^{(\alpha/2)}} W_u < a\right\}.$$

Clearly, η is a stopping time with respect to the natural filtration $\mathcal{F}_t$. As in the proof of Lemma 3.2, we have

$$\left\{\sup_{u\leq S_t^{(\alpha/2)}} W_u > b \text{ and } \inf_{u\leq S_t^{(\alpha/2)}} W_u < a\right\}$$
$$\subset \left\{\eta < t, \sup_{u\leq S_t^{(\alpha/2)}} \tilde{W}_u > b-a\right\} \cup \left\{\eta < t, \inf_{u\leq S_t^{(\alpha/2)}} \tilde{W}_u < -(b-a)\right\},$$

where $\tilde{W}_u := W_{u+\eta} - W_\eta$. By the strong Markov property, $\tilde{W}$ have the same distribution as W started from 0, and is independent of $\mathcal{F}_\eta$. Hence, using a similar argument that leads to (3.2), the symmetry of $\tilde{W}$, and Lemma 4.1, we have

$$\begin{aligned}
&\mathbb{P}_x\left(\sup_{u\le S_t^{(\alpha/2)}} W_u > b \text{ and } \inf_{u\le S_t^{(\alpha/2)}} W_u < a\right)\\
&\le 2\mathbb{P}_x(\eta<t)\mathbb{P}\left(\sup_{u\le S_t}\tilde{W}_u > b-a\right) \le c_1 t\mathbb{P}_x(\eta<t). \qquad (4.1)
\end{aligned}$$

When W starts at $x\in(a,b)$, we have

$$\{\eta<t\} = \left\{\sup_{u\le S_t^{(\alpha/2)}} W_u > b \text{ or } \inf_{u\le S_t^{(\alpha/2)}} W_u < a\right\}.$$

Hence, from Eq. 4.1, we have

$$\begin{aligned}
&\int_a^b \mathbb{P}_x\left(\sup_{u\le S_t^{(\alpha/2)}} W_u > b \text{ and } \inf_{u\le S_t^{(\alpha/2)}} W_u < a\right)dx\\
&\le c_1 t\int_a^b \mathbb{P}_x\left(\sup_{u\le S_t^{(\alpha/2)}} W_u > b \text{ or } \inf_{u\le S_t^{(\alpha/2)}} W_u < a\right)dx\\
&\le c_1 t\left(\int_a^b \mathbb{P}_x\left(\sup_{u\le S_t^{(\alpha/2)}} W_u > b\right)dx + \int_a^b \mathbb{P}_x\left(\inf_{u\le S_t^{(\alpha/2)}} W_u < a\right)dx\right). \qquad (4.2)
\end{aligned}$$

Note that it follows from the scaling property of $S^{(\alpha/2)}$ and W, independence of $S^{(\alpha/2)}$ and W, and the change of variable $y=t^{-1/\alpha}(b-x)$, we have

$$\begin{aligned}
&\int_a^b \mathbb{P}_x\left(\sup_{s\le S_t^{(\alpha/2)}} W_s \ge b\right)dx = \int_a^b \mathbb{P}\left(\sup_{s\le t^{2/\alpha}S_1^{(\alpha/2)}} t^{1/\alpha}W_{t^{-2/\alpha}s} \ge b-x\right)dx\\
&= t^{1/\alpha}\int_0^{(b-a)t^{-1/\alpha}} \mathbb{P}\left(\sup_{u\le S_1^{(\alpha/2)}} W_u \ge y\right)dy\\
&= t^{1/\alpha}\int_0^{(b-a)t^{-1/\alpha}}\int_0^\infty \mathbb{P}\left(\sup_{u\le v} W_u \ge y\right)g_1^{(\alpha/2)}(v)dvdy\\
&\le t^{1/\alpha}\int_0^\infty\int_0^\infty \mathbb{P}\left(\sup_{u\le v} W_u \ge y\right)dyg_1^{(\alpha/2)}(v)dv = t^{1/\alpha}\int_0^\infty \mathbb{E}\left[\overline{W}_v\right]g_1^{(\alpha/2)}(v)dv\\
&= t^{1/\alpha}\int_0^\infty v^{1/2}\mathbb{E}\left[\overline{W}_1\right]g_1^{(\alpha/2)}(v)dv = t^{1/\alpha}\mathbb{E}\left[\overline{W}_1\right]\mathbb{E}\left[\left(S_1^{(\alpha/2)}\right)^{1/2}\right]\\
&= t^{1/\alpha}\mathbb{E}\left[\overline{W}_{S_1^{(\alpha/2)}}\right], \qquad (4.3)
\end{aligned}$$

where the last term is known to be finite, since $\alpha > 1$ (see [1, Proposition 2.1]). By the symmetry of W, we similarly have

$$\int_a^b \mathbb{P}_x\left(\inf_{u\le S_t^{(\alpha/2)}} W_u < a\right)dx = \int_a^b \mathbb{P}\left(\inf_{s\le t^{2/\alpha}S_1^{(\alpha/2)}} t^{1/\alpha}W_{t^{-2/\alpha}s} < a-x\right)dx$$

$$= t^{1/\alpha}\int_{-(b-a)t^{-1/\alpha}}^0 \mathbb{P}\left(\inf_{u\le S_1^{(\alpha/2)}} W_u < y\right)dy = t^{1/\alpha}\int_0^{(b-a)t^{-1/\alpha}} \mathbb{P}\left(\sup_{u\le S_1^{(\alpha/2)}} W_u \ge y\right)dy$$

$$\le t^{1/\alpha}\mathbb{E}\left[\overline{W}_{S_1^{(\alpha/2)}}\right]. \tag{4.4}$$

Now the conclusion follows immediately from Eqs. 4.2, 4.3, and 4.4. □

Now, we reprove the following theorem using the probabilistic argument similar to [2].

Theorem 4.3 *Let $\alpha \in (1,2)$ and $D = (a,b)$ an open interval with $b-a < \infty$. Then, we have*

$$\lim_{t\to 0}\frac{|D| - \tilde{Q}_D^{(\alpha)}(t)}{t^{1/\alpha}} = \frac{2\Gamma\left(1-\frac{1}{\alpha}\right)}{\pi}|\partial D| = \mathbb{E}\left[\overline{W}_{S_1^{(\alpha/2)}}\right]|\partial D|.$$

Proof The proof is similar to [2, Theorem 1.1]. When W starts at $x \in (a,b)$, we have

$$\left\{\tau_D^{(2)} \le S_t^{(\alpha/2)}\right\} = \left\{W_s \ge b \text{ or } W_s \le a \text{ for some } s \le S_t^{(\alpha/2)}\right\}.$$

Hence, using a similar argument as (2.9), we have

$$|D| - \tilde{Q}_D^{(\alpha)}(t) = \int_D \mathbb{P}_x\left(\tau_D^{(2)} \le S_t^{(\alpha/2)}\right)dx = \int_D \mathbb{P}_x\left(\overline{W}_{S_t^{(\alpha/2)}} \ge b \text{ or } \underline{W}_{S_t^{(\alpha/2)}} \le a\right)dx$$

$$= \int_D \mathbb{P}_x\left(\overline{W}_{S_t^{(\alpha/2)}} \ge b\right) + \int_D \mathbb{P}_x\left(\underline{W}_{S_t^{(\alpha/2)}} \le a\right) - \int_D \mathbb{P}_x\left(\overline{W}_{S_t^{(\alpha/2)}} \ge b \text{ and } \underline{W}_{S_t^{(\alpha/2)}} \le a\right)dx. \tag{4.5}$$

From Lemma 4.2, the last expression above is $o(t^{1/\alpha})$. From Eqs. 4.3 and 4.4, and the monotone convergence theorem, we have

$$\lim_{t\to 0}\frac{\int_D \mathbb{P}_x\left(\overline{W}_{S_t^{(\alpha/2)}} \ge b\right)}{t^{1/\alpha}} = \lim_{t\to 0}\frac{\int_D \mathbb{P}_x\left(\underline{W}_{S_t^{(\alpha/2)}} \le a\right)}{t^{1/\alpha}} = \mathbb{E}\left[\overline{W}_{S_1^{(\alpha/2)}}\right].$$

Finally, from [1, Proposition 2.1], we have $\mathbb{E}\left[\left(S_1^{(\alpha/2)}\right)^{1/2}\right] = \frac{\Gamma\left(1-\frac{1}{\alpha}\right)}{\sqrt{\pi}}$ and from [10, Theorem 2.21] and a direct computation, we have

$$\mathbb{E}\left[\overline{W}_1\right] = \mathbb{E}\left[|W_1|\right] = 2\int_0^\infty \frac{x}{\sqrt{4\pi}}e^{-\frac{x^2}{4}}dx = \frac{2}{\sqrt{\pi}}.$$

From the independence of W and $S^{(\alpha/2)}$, this shows that

$$\mathbb{E}\left[\overline{W}_{S_1^{(\alpha/2)}}\right] = \mathbb{E}\left[\overline{W}_1\right]\mathbb{E}\left[\left(S_1^{(\alpha/2)}\right)^{1/2}\right] = \frac{2\Gamma\left(1-\frac{1}{\alpha}\right)}{\pi}.$$

□

Next, we need the following technical computations.

Lemma 4.4 *We have*

$$\mathbb{P}\left(\overline{W}_1 > u\right) \sim \frac{2}{\sqrt{\pi} u} e^{-\frac{u^2}{4}} \text{ as } u \to \infty.$$

Proof It follows from [10, Theorem 2.21] that we have

$$\mathbb{P}\left(\overline{W}_1 > u\right) = \mathbb{P}(|W_1| > u) = 2\int_u^\infty \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}} dx.$$

Now it follows from the L'Hôpital's rule, we have

$$\lim_{u\to\infty} \frac{2\int_u^\infty \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}} d}{\frac{2}{\sqrt{\pi} u} e^{-\frac{u^2}{4}}} = \lim_{u\to\infty} \frac{-e^{-\frac{u^2}{4}}}{-\frac{2}{u^2} e^{-\frac{u^2}{4}} - e^{-\frac{u^2}{4}}} = 1.$$

□

Lemma 4.5 *Let* $\alpha \in (1, 2)$. *Then, we have*

$$\lim_{t\to 0} \frac{\int_{(b-a)t^{-1/\alpha}}^\infty \int_0^{y^2} \mathbb{P}\left(\overline{W}_1 \geq \frac{y}{\sqrt{v}}\right) g_1^{(\alpha/2)}(v) dv dy}{t^{1-\frac{1}{\alpha}}}$$
$$= \frac{\alpha}{(\alpha-1)\Gamma\left(1-\frac{1}{\alpha}\right)(b-a)^{\alpha-1}} \int_1^\infty \mathbb{P}\left(\overline{W}_1 \geq u\right) u^{\alpha-1} du.$$

Proof By L'Hôpital's rule, the change of variable $u = \frac{(b-a)t^{-1/\alpha}}{\sqrt{v}}$, Eqs. 2.2 and 2.4, [12, Corollary 8.9], and the Lebesgue dominated convergence theorem using Lemma 4.4, we have

$$\lim_{t\to 0} \frac{\int_{(b-a)t^{-1/\alpha}}^\infty \int_0^{y^2} \mathbb{P}\left(\overline{W}_1 \geq \frac{y}{\sqrt{v}}\right) g_1^{(\alpha/2)}(v) dv dy}{t^{1-\frac{1}{\alpha}}}$$
$$= \lim_{t\to 0} \frac{(b-a)}{(\alpha-1)t} \int_0^{(b-a)^2 t^{-2/\alpha}} \mathbb{P}\left(\overline{W}_1 \geq \frac{(b-a)t^{-1/\alpha}}{\sqrt{v}}\right) g_1^{(\alpha/2)}(v) dv$$
$$= \lim_{t\to 0} \frac{2(b-a)^3}{(\alpha-1)t} \int_1^\infty \mathbb{P}\left(\overline{W}_1 \geq u\right) g_1^{(\alpha/2)}\left(\frac{(b-a)^2 t^{-2/\alpha}}{u^2}\right) t^{-2/\alpha} u^{-3} du$$
$$= \lim_{t\to 0} \frac{2(b-a)^3}{(\alpha-1)} \int_1^\infty \mathbb{P}\left(\overline{W}_1 \geq u\right) \frac{g_t^{(\alpha/2)}\left(\frac{(b-a)^2}{u^2}\right)}{t} u^{-3} du$$
$$= \frac{\alpha}{(\alpha-1)\Gamma\left(1-\frac{\alpha}{2}\right)(b-a)^{\alpha-1}} \int_1^\infty \mathbb{P}\left(\overline{W}_1 \geq u\right) u^{\alpha-1} du.$$

□

Recall that it follows from [11, Equation (2.5)],

$$\lim_{x\to\infty} g_1^{(\alpha/2)}(x) x^{1+\frac{\alpha}{2}} = \frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)}. \tag{4.6}$$

Lemma 4.6 *Let* $\alpha \in (1, 2)$. *Then, we have*

$$\lim_{t\to 0} \frac{\int_{(b-a)t^{-1/\alpha}}^\infty \int_{y^2}^\infty \mathbb{P}\left(\overline{W}_1 \geq \frac{y}{v^{1/2}}\right) g_1^{(\alpha/2)}(v) dv dy}{t^{1-\frac{1}{\alpha}}}$$
$$= \frac{\alpha}{(\alpha-1)\Gamma\left(1-\frac{\alpha}{2}\right)(b-a)^{\alpha-1}} \int_0^1 \mathbb{P}\left(\overline{W}_1 \geq w\right) w^{\alpha-1} dw.$$

Proof By the change of variable $w = \frac{y}{v^{1/2}}$, the inner integral in the numerator can be written as

$$\begin{aligned}\int_{y^2}^{\infty} \mathbb{P}\left(\overline{W}_1 \geq \frac{y}{v^{1/2}}\right) g_1^{(\alpha/2)}(v)dv &= \int_{y^2}^{\infty} \mathbb{P}\left(\overline{W}_1 \geq \frac{y}{v^{1/2}}\right) g_1^{(\alpha/2)}(v)dv \\ &= \int_0^1 \mathbb{P}\left(\overline{W}_1 \geq w\right) g_1^{(\alpha/2)}\left(\frac{y^2}{w^2}\right) \frac{2y^2}{w^3}dw.\end{aligned}$$

Since $g_1^{(\alpha/2)}(x) \leq cx^{-1-\frac{\alpha}{2}}$ for $x \geq 1$, the integral is finite.

By the L'Hôpital's rule, the Lebesgue dominated convergence theorem, and (4.6), we have

$$\begin{aligned}&\lim_{t\to 0} \frac{\int_{(b-a)t^{-1/\alpha}}^{\infty} \int_0^1 \mathbb{P}\left(\overline{W}_1 \geq w\right) g_1^{(\alpha/2)}\left(\frac{y^2}{w^2}\right) \frac{2y^2}{w^3}dwdy}{t^{1-\frac{1}{\alpha}}} \\ &= \lim_{t\to 0} \frac{2(b-a)}{(\alpha-1)t} \int_0^1 \mathbb{P}\left(\overline{W}_1 \geq w\right) g_1^{(\alpha/2)}\left(\left(\frac{(b-a)t^{-1/\alpha}}{w}\right)^2\right) \frac{\left((b-a)t^{-1/\alpha}\right)^2}{w^3}dw \\ &= \lim_{t\to 0} \frac{2}{(\alpha-1)(b-a)^{\alpha-1}} \int_0^1 \mathbb{P}\left(\overline{W}_1 \geq w\right) g_1^{(\alpha/2)}\left(\left(\frac{(b-a)t^{-1/\alpha}}{w}\right)^2\right) \\ &\quad \left(\frac{(b-a)t^{-1/\alpha}}{w}\right)^{2(1+\frac{\alpha}{2})} w^{\alpha-1}dw \\ &= \frac{2}{(\alpha-1)(b-a)^{\alpha-1}} \int_0^1 \mathbb{P}\left(\overline{W}_1 \geq w\right) \frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)} w^{\alpha-1}dw.\end{aligned}$$

□

Now we are ready to prove the first part of Theorem 1.2.

Proof of (1.3)

Note that from Eqs. 4.3, 4.4, and 4.5, we have

$$\begin{aligned}&|D| - \tilde{Q}_D^{(\alpha)}(t) \\ &= 2t^{1/\alpha} \int_0^{\infty} \int_0^{\frac{b-a}{t^{1/\alpha}}} \mathbb{P}\left(\sup_{u\leq v} W_u \geq y\right) dy g_1^{(\alpha/2)}(v)dv \\ &\quad - \int_D \mathbb{P}_x\left(\overline{W}_{S_t^{(\alpha/2)}} > b \text{ and } \underline{W}_{S_t^{(\alpha/2)}} < a\right) dx.\end{aligned} \tag{4.7}$$

It follows from Lemma 4.2

$$\int_D \mathbb{P}_x\left(\overline{W}_{S_t^{(\alpha/2)}} > b \text{ and } \underline{W}_{S_t^{(\alpha/2)}} < a\right) dx = O\left(t^{1+\frac{1}{\alpha}}\right).$$

Now we focus on the first integral in Eq. 4.7. Note that from the scaling property of W, we have

$$2t^{1/\alpha} \int_0^{\infty} \int_0^{\frac{b-a}{t^{1/\alpha}}} \mathbb{P}\left(\sup_{u\leq v} W_u \geq y\right) dy g_1^{(\alpha/2)}(v)dv - 2t^{1/\alpha} \int_0^{\infty} \mathbb{P}\left(\sup_{u\leq S_1^{(\alpha/2)}} W_u \geq y\right) dy$$

$$= 2t^{1/\alpha} \int_0^\infty \int_0^{\frac{b-a}{t^{1/\alpha}}} \mathbb{P}\left(\sup_{u \le v} W_u \ge y\right) dy g_1^{(\alpha/2)}(v) dv - 2t^{1/\alpha} \int_0^\infty \int_0^\infty \mathbb{P}\left(\sup_{u \le v} W_u \ge y\right) g_1^{(\alpha/2)}(v) dv dy$$

$$= -2t^{1/\alpha} \int_0^\infty \int_{\frac{b-a}{t^{1/\alpha}}}^\infty \mathbb{P}\left(\overline{W}_v \ge y\right) dy g_1^{(\alpha/2)}(v) dv = -2t^{1/\alpha} \int_0^\infty \int_{\frac{b-a}{t^{1/\alpha}}}^\infty \mathbb{P}\left(\overline{W}_1 \ge \frac{y}{\sqrt{v}}\right) dy g_1^{(\alpha/2)}(v) dv$$

$$= -2t^{1/\alpha} \left(\int_{\frac{b-a}{t^{1/\alpha}}}^\infty \int_0^{y^2} \mathbb{P}\left(\overline{W}_1 \ge \frac{y}{\sqrt{v}}\right) g_1^{(\alpha/2)}(v) dv dy + \int_{\frac{b-a}{t^{1/\alpha}}}^\infty \int_{y^2}^\infty \mathbb{P}\left(\overline{W}_1 \ge \frac{y}{\sqrt{v}}\right) g_1^{(\alpha/2)}(v) dv dy. \right.$$

Now the conclusion follows immediately from Lemmas 4.5 and 4.6. □

4.2 Case: $\alpha = 1$

In this subsection, we study the spectral heat content for subordinate killed Brownian motions when the underlying subordinator is $S^{(1/2)}$.

Proposition 4.7 *For any $u > 1$, we have*

$$\mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) = \frac{2}{\pi} \arctan(1/u). \tag{4.8}$$

Proof It follows from Eq. 2.3 that the density of $S_1^{(1/2)}$ is given by

$$g_1^{(1/2)}(x) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n+1} \frac{\Gamma\left(n + \frac{1}{2}\right)}{(2n-1)!} x^{-n-\frac{1}{2}}, \quad x > 0.$$

It is easy to check $(n!)^2 \le (2n-1)!$ for all $n \ge 1$ and $\Gamma\left(n + \frac{1}{2}\right) \le \Gamma(n+1)$ for all $n \ge 2$. Hence, we have

$$\left| \sum_{n=1}^\infty (-1)^{n+1} \frac{\Gamma\left(n + \frac{1}{2}\right)}{(2n-1)!} x^{-n-\frac{1}{2}} \right| \le \sum_{n=1}^\infty \frac{\Gamma\left(n + \frac{1}{2}\right)}{(2n-1)!} x^{-n-\frac{1}{2}} \le \Gamma\left(\frac{3}{2}\right) x^{-\frac{3}{2}} + \sum_{n=2}^\infty \frac{\Gamma(n+1)}{(2n-1)!} x^{-n-\frac{1}{2}}$$

$$\le \Gamma\left(\frac{3}{2}\right) x^{-\frac{3}{2}} + \sum_{n=2}^\infty \frac{x^{-n-\frac{1}{2}}}{n!} = \Gamma\left(\frac{3}{2}\right) x^{-\frac{3}{2}} + x^{-\frac{1}{2}} \left(e^{1/x} - 1 - \frac{1}{x}\right)$$

$$= O\left(\frac{1}{x^{3/2}}\right) \text{ as } x \to \infty.$$

Hence, by the Lebesgue dominated convergence theorem, we have

$$\mathbb{P}\left(S_1^{(1/2)} > \frac{u^2}{x^2}\right) = \int_{\frac{u^2}{x^2}}^{\infty} g_1^{(1/2)}(v)dv$$

$$= \int_{\frac{u^2}{x^2}}^{\infty} \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma\left(n+\frac{1}{2}\right)}{(2n-1)!} v^{-n-\frac{1}{2}} dv = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma\left(n+\frac{1}{2}\right)}{(2n-1)!} \frac{1}{n-\frac{1}{2}} \frac{x^{2n-1}}{u^{2n-1}}.$$

It follows from [10, Theorem 2.21] that

$$\mathbb{P}\left(\overline{W}_1 > a\right) = \mathbb{P}(|W_1| > a) = 2\int_a^{\infty} \frac{1}{\sqrt{4\pi}} \int_a^{\infty} e^{-\frac{x^2}{4}} dx = \frac{1}{\sqrt{\pi}} \int_a^{\infty} e^{-\frac{x^2}{4}} dx.$$

Hence, we have

$$\mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) = \int_0^{\infty} \mathbb{P}\left(\overline{W}_y > u\right) g_1^{(\alpha/2)}(y)dy = \int_0^{\infty} \mathbb{P}\left(\overline{W}_1 > \frac{u}{\sqrt{y}}\right) g_1^{(\alpha/2)}(y)dy$$

$$= \int_0^{\infty} \frac{1}{\sqrt{\pi}} \int_{\frac{u}{\sqrt{y}}}^{\infty} e^{-\frac{x^2}{4}} dx g_1^{(\alpha/2)}(y)dy = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \int_{\frac{u^2}{x^2}}^{\infty} g_1^{(\alpha/2)}(y)dy e^{-\frac{x^2}{4}} dx$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} \left(\frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma\left(n+\frac{1}{2}\right)}{(2n-1)!} \frac{1}{n-\frac{1}{2}} \frac{x^{2n-1}}{u^{2n-1}}\right) e^{-\frac{x^2}{4}} dx$$

$$= \frac{2}{\pi^{3/2}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma\left(n+\frac{1}{2}\right)}{(2n-1)!} \frac{1}{2n-1} \frac{1}{u^{2n-1}} \int_0^{\infty} x^{2n-1} e^{-\frac{x^2}{4}} dx$$

$$= \frac{2}{\pi^{3/2}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma\left(n+\frac{1}{2}\right)}{(2n-1)!} \frac{1}{2n-1} \frac{1}{u^{2n-1}} \Gamma(n) 2^{2n-1}, \tag{4.9}$$

where we used $\int_0^{\infty} x^{2n-1} e^{-\frac{x^2}{4}} dx = \Gamma(n)2^{2n-1}$, and the interchange of the infinite sum and integral is valid, because of the exponential decay term and the fact $u > 1$. By the Legendre duplication formula, we have $\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = 2^{1-2n}\sqrt{\pi}\Gamma(2n)$. By the Taylor expansion of $\arctan(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$ for $|x| < 1$, Eq. 4.9 can be simplified to

$$\frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1} \frac{1}{u^{2n-1}} = \frac{2}{\pi} \arctan\left(\frac{1}{u}\right), \quad \text{for } u > 1.$$

□

Remark 4.8 Even though it is not necessary for our result, it would be interesting to see if (4.8) holds for all $u > 0$.

Lemma 4.9 *We have*

$$\int_a^b \mathbb{P}_x\left(\overline{W}_{S_t^{(1/2)}} > b \text{ and } \underline{W}_{S_t^{(1/2)}} < a\right) dx = O(t^2 \ln(1/t)).$$

Proof The proof is almost identical to the proof of Lemmas 3.3 using 4.1. It follows from Proposition 4.7, $\mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) \sim \frac{2}{\pi u}$ as $u \to \infty$, and this shows that $\int_0^{\frac{b-a}{t}} \mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) du = O(\ln(1/t))$. □

Now we are ready to prove the second part of Theorem 1.2.

Proof of (1.4)
Note that from Eq. 4.5, we have

$$|D| - \tilde{Q}_D^{(1)}(t) = \int_a^b \mathbb{P}_x\left(\tau_D^{(2)} \le S_t^{(1/2)}\right) dx$$
$$= 2\int_a^b \mathbb{P}_x\left(\overline{W}_{S_t^{(1/2)}} > b\right) dx - \int_a^b \mathbb{P}_x\left(\overline{W}_{S_t^{(1/2)}} > b \text{ and } \underline{W}_{S_t^{(1/2)}} < a\right) dx.$$

It follows from Lemma 4.9, the second term is $O\left(t^2 \ln(1/t)\right)$.
Now the first expression above can be written as

$$2\int_a^b \mathbb{P}_x\left(\overline{W}_{S_t^{(1/2)}} > b\right) dx = 2\int_a^b \mathbb{P}_x\left(\sup_{u \le t^2 S_1^{(1/2)}} tW_{ut^{-2}} > b\right) dx$$
$$= 2\int_a^b \mathbb{P}_x\left(\sup_{v \le S_1^{(1/2)}} W_v > b/t\right) dx = 2t\int_0^{\frac{b-a}{t}} \mathbb{P}\left(\sup_{v \le S_1^{(1/2)}} W_v > u\right) du$$
$$= 2t\int_0^{\frac{b-a}{t}} \mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) du.$$

Hence, we have

$$2t\int_0^{\frac{b-a}{t}} \mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) du - \frac{4}{\pi} t \ln(1/t)$$
$$= 2t\left(\int_0^{\frac{b-a}{t}} \mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) du - \frac{2}{\pi}\ln(1/t)\right)$$
$$= 2t\left(\int_0^1 \mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) du + \frac{2\ln(b-a)}{\pi} + \int_1^{\frac{b-a}{t}} \mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) - \frac{2}{\pi u} du\right).$$

From Proposition 4.7, we have $\mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) - \frac{2}{\pi u} = O\left(\frac{1}{u^3}\right)$, and this shows that it is integrable on $(1, \infty)$. Hence, it follows from the monotone convergence theorem

$$\lim_{t\to 0} \frac{2t}{t}\left(\int_0^1 \mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) du + \frac{2\ln(b-a)}{\pi} + \int_1^{\frac{b-a}{t}} \mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) - \frac{2}{\pi u} du\right)$$
$$= 2\left(\int_0^1 \mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) du + \frac{2\ln(b-a)}{\pi} + \int_1^{\infty} \mathbb{P}\left(\overline{W}_{S_1^{(1/2)}} > u\right) - \frac{2}{\pi u} du\right).$$

□

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