Received: 23 January 2018 | Revised: 29 May 2018 | Accepted: 29 June 2018
DOI: 10.1002/mana.201800035

ORIGINAL PAPER

MATHEMATISCHE NACHRICHTEN

Spectral heat content for Lévy processes

Tomasz Grzywny[1] | Hyunchul Park[2] | Renming Song[3]

[1]Wydział Matematyki Politechnika, Wrocławska Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland

[2]Department of Mathematics, State University of New York at New Paltz, NY 12561, USA

[3]Department of Mathematics, University of Illinois, Urbana, IL 61801, USA

Correspondence
Hyunchul Park, Department of Mathematics, State University of New York at New Paltz, NY 12561, USA.
Email: parkh@newpaltz.edu

Funding information
National Science Centre (Poland), Grant/Award Number: 2016/23/B/ST1/01665; Simons Foundation, Grant/Award Numbers: 429343, Renming Song

Abstract
In this paper we study the spectral heat content for various Lévy processes. We establish the small time asymptotic behavior of the spectral heat content for Lévy processes of bounded variation in $\mathbb{R}^d$, $d \geq 1$. We also study the spectral heat content for arbitrary open sets of finite Lebesgue measure in $\mathbb{R}$ with respect to symmetric Lévy processes of unbounded variation under certain conditions on their characteristic exponents. Finally, we establish that the small time asymptotic behavior of the spectral heat content is stable under integrable perturbations to the Lévy measure.

KEYWORDS
heat content, infinitesimal generator, Lévy process, spectral heat content

MSC (2010)
35K05, 60G51, 60J75

1 | INTRODUCTION

Let $\mathbf{X} = (X_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$. For any open set $\Omega \subset \mathbb{R}^d$, the (regular) heat content of Ω with respect to $\mathbf{X}$ is defined to be

$$H_\Omega(t) = \int_\Omega \mathbb{P}_x(X_t \in \Omega)\,\mathrm{d}x,$$

while the spectral heat content of Ω with respect to $\mathbf{X}$ is defined to be

$$Q_\Omega(t) = \int_\Omega \mathbb{P}_x\big(\tau_\Omega > t\big)\mathrm{d}x,$$

where $\tau_\Omega = \inf\{t > 0 : X_t \in \Omega^c\}$ is the first time the process $\mathbf{X}$ exits Ω.

The asymptotic behaviors of the heat content and the spectral heat content have been studied intensively in the case of Brownian motion, see [22] and [27]–[32]. Recently significant progress has also been made in studying the heat content and the spectral heat content with respect to Lévy processes with discontinuous sample paths, see [1–3,8,19]. The asymptotic behaviors of the heat content and the spectral heat content with respect to symmetric stable processes were studied in [1–3]. In particular, the exact asymptotic behavior of the spectral heat content of bounded open intervals with respect to symmetric stable processes in $\mathbb{R}$ was established in [3]. The asymptotic behavior of the heat content with respect to general Lévy processes was studied in [8] (see also [9] for a generalization). In [19], an asymptotic expansion of the heat content with respect to some isotropic compound Poisson processes with compactly supported jumping kernels was established.

The purpose of this paper is to investigate the small time asymptotic behavior of the spectral heat content of general Lévy processes and generalize the results of [3] in several directions.

The organization of this paper is as follows. In Section 2 we recall some notions and present some preliminaries. In Section 3 we first study the heat content and the spectral heat content with respect to Lévy processes of bounded variation. In Theorem 3.2, we extend [8, Theorem 3] to any open set of finite Lebesgue measure and relax the finite perimeter condition. Then we use this to establish the small time asymptotic behavior of the spectral heat content for the same processes in Theorems 3.3 and 3.4. We remark here that when the underlying processes are of bounded variation, the small time asymptotic behavior of the spectral heat contents depends not only on the geometry of the open sets but also on the underlying processes (see Corollaries 3.5, 3.6, and 3.7). In Section 4 we investigate the small time asymptotic behavior of the spectral heat content with respect to symmetric Lévy processes of unbounded variation in $\mathbb{R}$. In this section we deal with two cases separately. In the first case, we assume that the characteristic exponent $\psi(\xi)$ of $\mathbf{X}$ is regularly varying of index α at infinity for some $\alpha \in (1,2]$. In the second case, we assume that $\mathbf{X}$ is a symmetric 1-stable process, that is, a Cauchy process. The main results in Section 4 are Theorems 4.2 and 4.14, where we establish the exact small time asymptotic behavior of the spectral heat content with respect to such processes. We note here that the small time asymptotic behavior of $Q_\Omega(t)$ depends on the geometry of Ω. When $\alpha \in (1,2]$, both the number of adjacent components and the number of non-adjacent components matter, while in the case $\alpha = 1$, only the number of non-adjacent components matters since the process can not hit a single point upon exiting the open set. Two components of Ω are said to be adjacent if the distance between them is zero. In Section 5 we study the stability of the spectral heat content. We prove in Theorem 5.1 that the small time asymptotic behavior of the spectral heat content is stable under integrable perturbations to the Lévy measures. In Section 6 we give some examples where one can apply the results of this paper to get the small time asymptotic behavior of the spectral heat content.

2 | PRELIMINARIES

Let $\mathbf{X} = (X_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$. We denote by (P_t) the semigroup of $\mathbf{X}$ and by $\widehat{P}_t$ the adjoint operator of P_t. The characteristic exponent $\psi(\xi)$, $\xi \in \mathbb{R}^d$, of $\mathbf{X}$ is given by

$$\psi(\xi) = \langle \xi, A\xi \rangle - i\langle \xi, \gamma \rangle - \int_{\mathbb{R}^d} \left(e^{i\langle \xi, y\rangle} - 1 - i\langle \xi, y\rangle 1_{\{\|y\|\leq 1\}} \right) \nu(\mathrm{d}y),$$

where A is a symmetric non-negative definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a Lévy measure, that is

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} \min\left\{1, \|y\|^2\right\} \nu(\mathrm{d}y) < \infty.$$

The Lévy process $\mathbf{X}$ is of bounded variation (see [24, Theorem 21.9]) if and only if

$$A = 0 \quad \text{and} \quad \int_{\|y\|\leq 1} \|y\| \, \nu(\mathrm{d}y) < \infty.$$

In this case the characteristic exponent has the following simple form

$$\psi(\xi) = i\langle \xi, \gamma_0 \rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, y\rangle}\right) \nu(\mathrm{d}y),$$

where $\gamma_0 = \int_{\|y\|\leq 1} y \, \nu(\mathrm{d}y) - \gamma$. For a Lévy process of bounded variation, the quantity γ_0 defined above is called the drift of the process.

For any open set Ω in $\mathbb{R}^d$, the killed process $\mathbf{X}_t^\Omega$ is defined by

$$\mathbf{X}_t^\Omega = \begin{cases} \mathbf{X}_t & \text{if } t < \tau_\Omega, \\ \partial & \text{if } t \geq \tau_\Omega, \end{cases}$$

where ∂ is a cemetery point and $\tau_\Omega = \inf\left\{t > 0 : \mathbf{X}_t \notin \Omega\right\}$ is the first exit time of $\mathbf{X}$ from Ω. The process $\mathbf{X}_t^\Omega$ is a strong Markov process and its semigroup is given by

$$P_t^\Omega f(x) = \mathbb{E}_x\left[f(X_t); t < \tau_\Omega\right], \quad x \in \Omega.$$

We introduce the following function related to $\mathbf{X}$, see [23]. For any $r > 0$,

$$h(r) = \|A\| r^{-2} + r^{-1} \left| \gamma + \int_{\mathbb{R}^d} y\left(1_{\{\|y\|<r\}} - 1_{\{\|y\|<1\}}\right) \nu(\mathrm{d}y) \right| + \int_{\mathbb{R}^d} \min\left\{1, \|y\|^2 r^{-2}\right\} \nu(\mathrm{d}y).$$

Recall that there exists $C_1 = C_1(d) > 0$, (see [23, page 941]) such that

$$\mathbb{P}\left(\sup_{s\le t}|X_s| > r\right) \le C_1 t h(r). \tag{2.1}$$

Following [4, Section 3.3], for any Borel set $\Omega \subset \mathbb{R}^d$, we define its perimeter $\mathrm{Per}(\Omega)$ as

$$\mathrm{Per}(\Omega) = \sup\left\{\int_{\mathbb{R}^d} 1_\Omega(x)\,\mathrm{div}\,\phi(x)\,\mathrm{d}x \,:\, \phi \in C_c^1\left(\mathbb{R}^d, \mathbb{R}^d\right),\ \|\phi\|_\infty \le 1\right\}.$$

We say that Ω is of finite perimeter if $\mathrm{Per}(\Omega) < \infty$. It was shown [20–22] that if $\Omega \subset \mathbb{R}^d$ is an open set of finite Lebesgue measure and of finite perimeter, then

$$\mathrm{Per}(\Omega) = \pi^{1/2}\lim_{t\to 0} t^{-1/2}\int_\Omega\int_{\Omega^c} p_t^{(2)}(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

where

$$p_t^{(2)}(x,y) = (4\pi t)^{-d/2} e^{-\|x-y\|^2/4t}$$

is the transition density of the Brownian motion $\mathbf{B} = (B_t)_{t\ge 0}$ in $\mathbb{R}^d$. For a Lévy process $\mathbf{X}$ with Lévy measure ν, we define the perimeter $\mathrm{Per}_{\mathbf{X}}(\Omega)$ with respect to $\mathbf{X}$ as

$$\mathrm{Per}_{\mathbf{X}}(\Omega) = \int_\Omega\int_{\Omega^c - x}\nu(\mathrm{d}y)\,\mathrm{d}x.$$

In particular, to the isotropic (rotationally invariant) α-stable process $\mathbf{S}^{(\alpha)} = \left(S_t^{(\alpha)}\right)_{t\ge 0}$, $0 < \alpha < 1$, one associates the α-perimeter which is defined by

$$\mathrm{Per}_{\mathbf{S}^{(\alpha)}}(\Omega) := \int_\Omega\int_{\Omega^c}\frac{c(d,\alpha)\,\mathrm{d}y\,\mathrm{d}x}{\|x-y\|^{d+\alpha}},$$

where

$$c(d,\alpha) := \frac{\alpha\,\Gamma\left(\frac{d+\alpha}{2}\right)}{2^{1-\alpha}\,\pi^{d/2}\Gamma\left(1-\frac{\alpha}{2}\right)}.$$

It is known, cf. [8, Lemma 1] (see also [13] for the perimeter for the isotropic stable processes), that if $\mathbf{X}$ is of bounded variation and $\Omega \subset \mathbb{R}^d$ is an open set of finite Lebesgue measure and of finite perimeter, then $\mathrm{Per}_{\mathbf{X}}(\Omega)$ is also finite.

Let $G \subset \mathbb{R}^d$ be an open set and let $f : G \to \mathbb{R}$ be integrable. The total variation of f in G is

$$V(f,G) = \sup\left\{\int_G f(x)\,\mathrm{div}\varphi(x)\,\mathrm{d}x \,:\, \varphi \in C_c^1\left(G, \mathbb{R}^d\right), \|\varphi\|_\infty \le 1\right\}.$$

The directional derivative of f in G in the direction of $u \in \mathbb{S}^{d-1}$ is

$$V_u(f,G) = \sup\left\{\int_G f(x)\langle\nabla\varphi(x), u\rangle\,\mathrm{d}x \,:\, \varphi \in C_c^1\left(G, \mathbb{R}^d\right), \|\varphi\|_\infty \le 1\right\}.$$

We will use $V_u(\Omega)$ to denote $V_u\left(1_\Omega, \mathbb{R}^d\right)$.

Now we recall the covariogram function $g_\Omega(y)$. Let

$$g_\Omega(y) = |\Omega \cap (\Omega + y)| = \int_{\mathbb{R}^d} 1_\Omega(x)1_{\Omega+y}(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} 1_\Omega(x)1_\Omega(x-y)\,\mathrm{d}x.$$

It is easy to see that

$$g_\Omega(y) \le g_\Omega(0) = |\Omega| \quad \text{and} \quad g_\Omega(-y) = g_\Omega(y).$$

Moreover $g_\Omega \in C_0(\mathbb{R}^d)$ (see [14, Proposition 2]).

Let

$$f_\Omega(y) := g_\Omega(0) - g_\Omega(y).$$

By the Fubini–Tonelli theorem we have the following relationship between $\mathrm{Per}_{\mathbf{X}}(\Omega)$ and $f_\Omega(\cdot)$

$$\mathrm{Per}_{\mathbf{X}}(\Omega) = \int_{\mathbb{R}^d} f_\Omega(y)\, \nu(\mathrm{d}y). \tag{2.2}$$

Here is a simple lemma about the behavior of $f_\Omega(y)$ as $|y| \to 0$.

Lemma 2.1. *If $\Omega \subset \mathbb{R}^d$ is an open set of finite Lebesgue measure $|\Omega|$, then*

$$\lim_{|y|\to 0} f_\Omega(y) = 0.$$

Proof. Note that $1_\Omega(x)1_{\Omega^c}(x-y) \le 1_\Omega(x)$ for all $y \in \mathbb{R}^d$ and $\int 1_\Omega(x)\,\mathrm{d}x = |\Omega| < \infty$. For each $x \in \Omega$ we have

$$\lim_{|y|\to 0} 1_\Omega(x)1_{\Omega^c}(x-y) = 0.$$

Hence the assertion follows from the dominated convergence theorem. □

The next lemma follows from [14, Proposition 5].

Lemma 2.2. *If $\Omega \subset \mathbb{R}^d$ is a Borel set of finite Lebesgue measure, then*

$$\left|g_\Omega(x) - g_\Omega(y)\right| \le g_\Omega(0) - g_\Omega(x-y), \quad x, y \in \mathbb{R}^d.$$

We end this section by recalling the concept of regularly varying functions. A function f is said to be regularly varying of index α at infinity if for any $\lambda > 0$,

$$\lim_{r\to\infty} \frac{f(\lambda r)}{f(r)} = \lambda^\alpha.$$

The family of regularly varying functions of index α at infinity is denoted by $\mathcal{R}_\alpha$.

3 | PROCESSES OF BOUNDED VARIATION IN $\mathbb{R}^d$

In this section we assume that $\mathbf{X}$ is a Lévy process of bounded variation in $\mathbb{R}^d$.

3.1 | Heat content

In this subsection, we first extend [8, Theorem 3] when the drift $\gamma_0 = 0$. Suppose that $\mathbf{X}$ is a purely discontinuous Lévy process of bounded variation in $\mathbb{R}^d$, that is, $A = 0$, $\gamma_0 = 0$ and $\int_{\|x\|\le 1} \|x\|\, \nu(\mathrm{d}x) < \infty$.

The infinitesimal generator $\mathcal{L}$ on $C_0(\mathbb{R}^d)$ of $\mathbf{X}$ is a linear operator defined by

$$\mathcal{L}f(x) = \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t}, \tag{3.1}$$

with domain $\mathrm{Dom}(\mathcal{L})$ consisting of all functions f such that the right hand side of (3.1) exists. By [24, Theorem 31.5], we have $C_0^2(\mathbb{R}^d) \subset \mathrm{Dom}(\mathcal{L})$. For a detailed discussion on infinitesimal generators of Lévy processes we refer the reader to [24, Section 31]. Since $\mathbf{X}$ is of bounded variation and $\gamma_0 = 0$, again by [24, Theorem 31.5] we have, for any $f \in C_0^2(\mathbb{R}^d)$,

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(x+y) - f(x))\, \nu(\mathrm{d}y). \tag{3.2}$$

The next lemma corresponds to [8, Lemma 2].

Lemma 3.1. *Suppose that $\Omega \subset \mathbb{R}^d$ is an open set of finite Lebesgue measure. If*

$$\mathrm{Per}_{\mathbf{X}}(\Omega) = \int_{\mathbb{R}^d} f_\Omega(y)\, \nu(\mathrm{d}y) < \infty,$$

then $g_\Omega(x) \in \mathrm{Dom}(\mathcal{L})$ and

$$\mathcal{L} g_\Omega(x) = \int_{\mathbb{R}^d} \big(g_\Omega(x+y) - g_\Omega(x)\big)\, \nu(\mathrm{d}y).$$

Proof. Fix a cut-off function $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi \geq 0$, $\operatorname{supp}\phi \subset B(0,1)$ and $\|\phi\|_1 = 1$. Let $\phi_\varepsilon(x) := \frac{1}{\varepsilon^d}\phi\big(\frac{x}{\varepsilon}\big)$ and $g_\Omega^\varepsilon(x) := g_\Omega * \phi_\varepsilon(x)$. Since g_Ω is integrable, we get that g_Ω^ε is smooth and vanishes at infinity. Hence $g_\Omega^\varepsilon \in C_0^\infty(\mathbb{R}^d)$. Since $C_0^2(\mathbb{R}^d) \subset \mathrm{Dom}(\mathcal{L})$, we get $g_\Omega^\varepsilon \in \mathrm{Dom}(\mathcal{L})$. Note that it follows from Lemma 2.2 that

$$|g_\Omega^\varepsilon(x) - g_\Omega(x)| = \left|\int_{\mathbb{R}^d} \phi_\varepsilon(y)\big(g_\Omega(x-y) - g_\Omega(x)\big)\,\mathrm{d}y\right| \leq \sup_{|y|\leq\varepsilon} |g_\Omega(0) - g_\Omega(y)|\|\phi_\varepsilon\|_1 = \sup_{|y|\leq\varepsilon} |f_\Omega(y)|.$$

Lemma 2.1 implies

$$\lim_{\varepsilon\downarrow 0} ||g_\Omega^\varepsilon - g_\Omega||_\infty = 0.$$

By (3.2),

$$\mathcal{L} g_\Omega^\varepsilon(x) = \int_{\mathbb{R}^d} \big(g_\Omega^\varepsilon(x+y) - g_\Omega^\varepsilon(x)\big)\, \nu(\mathrm{d}y).$$

Note that from Lemma 2.2 we have

$$|g_\Omega^\varepsilon(x+y) - g_\Omega^\varepsilon(x)| \leq f_\Omega(y), \quad x, y \in \mathbb{R}^d.$$

Hence, by the dominated convergence theorem we infer that

$$\lim_{\varepsilon\downarrow 0}\left\|\mathcal{L} g_\Omega^\varepsilon - \int_{\mathbb{R}^d} \big(g_\Omega(\cdot + y) - g_\Omega(\cdot)\big)\, \nu(\mathrm{d}y)\right\|_\infty = 0.$$

Since $\mathcal{L}$ is a closed operator, the assertion of the lemma is now established. □

The following result is similar in spirit to [8, Theorem 3]. The difference is that in the result below we do not assume $\mathrm{Per}(\Omega) < \infty$ but we assume that $\gamma_0 = 0$.

Theorem 3.2. *Let $\mathbf{X}$ be a Lévy process of bounded variation with $\gamma_0 = 0$. If $\Omega \subset \mathbb{R}^d$ is an open set of finite Lebesgue measure, then we have*

$$\lim_{t\to 0} \frac{|\Omega| - H_\Omega(t)}{t} = \mathrm{Per}_{\mathbf{X}}(\Omega).$$

Proof. First we assume that

$$\mathrm{Per}_{\mathbf{X}}(\Omega) = \int_\Omega \nu(\Omega^c - y)\,\mathrm{d}y < \infty.$$

In this case the proof is similar to that of [8, Theorem 3]. It follows from Lemma 3.1 and (2.2) that

$$\begin{aligned}\lim_{t\to 0} \frac{|\Omega| - H_\Omega(t)}{t} &= \lim_{t\to 0} \int_\Omega \frac{1 - \mathbb{P}(x + X_t \in \Omega)}{t}\,\mathrm{d}x \\ &= \lim_{t\to 0} \frac{g_\Omega(0) - P_t g_\Omega(0)}{t} = -\mathcal{L} g_\Omega(0) = \int_{\mathbb{R}^d} f_\Omega(y)\,\nu(\mathrm{d}y) = \mathrm{Per}_{\mathbf{X}}(\Omega).\end{aligned}$$

Now we deal with the case when

$$\int_{\Omega} \nu(\Omega^c - y)\,\mathrm{d}y = \infty.$$

Note that

$$\frac{|\Omega| - H_\Omega(t)}{t} = \int_{\mathbb{R}^d} f_\Omega(y)\,\frac{p_t(\mathrm{d}y)}{t},$$

where $p_t(\mathrm{d}y)$ corresponds to the transition density of $\mathbf{X}$ started from the origin. Let $\varepsilon > 0$, and $\phi_\varepsilon \in C_b(\mathbb{R}^d)$ be such that $1_{B(0,\varepsilon)^c} \le \phi_\varepsilon \le 1_{B(0,\varepsilon/2)^c}$. Then by [24, Corollary 8.9] we get

$$\liminf_{t\to 0} \frac{|\Omega| - H_\Omega(t)}{t} \ge \liminf_{t\to 0} \int_{\mathbb{R}^d} \phi_\varepsilon(y) f_\Omega(y)\,\frac{p_t(\mathrm{d}y)}{t} = \int_{\mathbb{R}^d} \phi_\varepsilon(y) f_\Omega(y)\,\nu(\mathrm{d}y) \ge \int_{B(0,\varepsilon)^c} f_\Omega(y)\,\nu(\mathrm{d}y).$$

Since ε is arbitrary, we have

$$\liminf_{t\to 0} \frac{|\Omega| - H_\Omega(t)}{t} = \infty. \qquad \square$$

3.2 | Spectral heat content

In this subsection we study the small time asymptotic behavior of the spectral heat content for Lévy processes of bounded variation. The main result is the following theorem.

Theorem 3.3. *Let $\mathbf{X}$ be a Lévy process of bounded variation in $\mathbb{R}^d$. If $\Omega \subset \mathbb{R}^d$ is an open set of finite measure $|\Omega|$ and of finite perimeter* $\mathrm{Per}(\Omega)$*, then*

$$\begin{aligned}
\limsup_{t\to 0} \frac{|\Omega| - Q_\Omega(t)}{t} &= \mathrm{Per}_{\mathbf{X}}(\Omega) + \frac{\|\gamma_0\|}{2} V_{\frac{\gamma_0}{\|\gamma_0\|}}(\Omega)\mathbf{1}_{\mathbb{R}^d\setminus\{0\}}(\gamma_0)\\
&\quad + \limsup_{t\to 0} \frac{1}{t}\int_\Omega \mathbb{E}_x\Big[\tau_\Omega < t, X_{\tau_\Omega} \in \partial\Omega,\ \mathbb{P}_{X_{\tau_\Omega}}\big(X_{t-\tau_\Omega} \in \Omega\big)\Big]\,\mathrm{d}x,\\
\liminf_{t\to 0} \frac{|\Omega| - Q_\Omega(t)}{t} &= \mathrm{Per}_{\mathbf{X}}(\Omega) + \frac{\|\gamma_0\|}{2} V_{\frac{\gamma_0}{\|\gamma_0\|}}(\Omega)\mathbf{1}_{\mathbb{R}^d\setminus\{0\}}(\gamma_0)\\
&\quad + \liminf_{t\to 0} \frac{1}{t}\int_\Omega \mathbb{E}_x\Big[\tau_\Omega < t, X_{\tau_\Omega} \in \partial\Omega,\ \mathbb{P}_{X_{\tau_\Omega}}\big(X_{t-\tau_\Omega} \in \Omega\big)\Big]\,\mathrm{d}x,
\end{aligned}$$

where $V_u(\Omega)$ is the directional derivative of 1_Ω in the direction u on the unit sphere in $\mathbb{R}^d$.

Proof. By the right continuity of paths of $\mathbf{X}$ we obtain

$$\mathbb{P}_x(\tau_\Omega > t) = \mathbb{P}_x\big(X_t \in \Omega, \tau_\Omega > t\big) = \mathbb{P}_x(X_t \in \Omega) - \mathbb{P}_x\big(X_t \in \Omega, \tau_\Omega < t\big).$$

Hence by applying strong Markov property at τ_Ω we have

$$\begin{aligned}
|\Omega| - Q_\Omega(t) &= \int_\Omega \big(1 - \mathbb{P}_x(X_t \in \Omega)\big)\,\mathrm{d}x + \int_\Omega \mathbb{E}_x\Big[\tau_\Omega < t, \mathbb{P}_{X_{\tau_\Omega}}\big(X_{t-\tau_\Omega} \in \Omega\big)\Big]\,\mathrm{d}x\\
&= \big(|\Omega| - H_\Omega(t)\big) + \int_\Omega \mathbb{E}_x\Big[\tau_\Omega < t, \mathbb{P}_{X_{\tau_\Omega}}\big(X_{t-\tau_\Omega} \in \Omega\big)\Big]\,\mathrm{d}x\\
&= \big(|\Omega| - H_\Omega(t)\big) + \mathrm{I}(t) + \mathrm{II}(t),
\end{aligned}$$

where

$$\mathrm{I}(t) = \int_\Omega \mathbb{E}_x\Big[\tau_\Omega < t, X_{\tau_\Omega} \in \overline{\Omega}^c, \mathbb{P}_{X_{\tau_\Omega}}\big(X_{t-\tau_\Omega} \in \Omega\big)\Big]\,\mathrm{d}x,$$

$$\mathrm{II}(t) = \int_\Omega \mathbb{E}_x\Big[\tau_\Omega < t, X_{\tau_\Omega} \in \partial\Omega, \mathbb{P}_{X_{\tau_\Omega}}\big(X_{t-\tau_\Omega} \in \Omega\big)\Big]\,\mathrm{d}x.$$

By [8, Theorem 3], it suffices to show that

$$\lim_{t\to 0}\frac{\mathrm{I}(t)}{t}=0.$$

By the Ikeda–Watanabe formula [18], the joint distribution of $\left(\tau_\Omega, X_{\tau_\Omega}\right)$ restricted to $X_{\tau_\Omega-}\neq X_{\tau_\Omega}$ is equal to

$$\mathbb{P}_x\left(\left(\tau_\Omega, X_{\tau_\Omega}\right)\in(\mathrm{d}s,\mathrm{d}z)\right)=\int_\Omega p_s^\Omega(x,\mathrm{d}u)\,\nu(\mathrm{d}z-u)\,\mathrm{d}s,$$

where $p_s^\Omega(x,\mathrm{d}u)$ is the transition kernel of the process $\mathbf{X}$ killed upon exiting Ω. Hence

$$\mathrm{I}(t)=\int_\Omega \mathrm{d}x\int_0^t \mathrm{d}s\int_{\overline{\Omega}^c}\mathbb{P}_z(X_{t-s}\in\Omega)\int_\Omega p_s^\Omega(x,\mathrm{d}u)\,\nu(\mathrm{d}z-u)\leq\int_0^t\mathrm{d}s\int_\Omega P_s(g_{t-s})(x)\,\mathrm{d}x,$$

where

$$g_s(u)=\mathbf{1}_\Omega(u)\int_{\overline{\Omega}^c-u}\mathbb{P}_{z+u}(X_s\in\Omega)\,\nu(\mathrm{d}z).$$

Notice that (see [8, Lemma 1])

$$\int_\Omega g_s(x)\,\mathrm{d}x\leq\int_\Omega\int_{\Omega^c-x}\nu(\mathrm{d}z)\,\mathrm{d}x=\mathrm{Per}_{\mathbf{X}}(\Omega)<\infty. \tag{3.3}$$

Thus

$$\mathrm{I}(t)\leq\int_0^t\mathrm{d}s\int_\Omega\hat{P}_s\left(\mathbf{1}_\Omega\right)(x)g_{t-s}(x)\,\mathrm{d}x\leq\int_0^t\mathrm{d}s\int_\Omega g_s(x)\,\mathrm{d}x.$$

By the right continuity of $\mathbf{X}$, we have $\lim_{t\to 0}\mathbb{P}_y(X_t\in\Omega)=0$ for all $y\in\overline{\Omega}^c$. Thus by the dominated convergence theorem and (3.3),

$$\limsup_{t\to 0}\frac{\mathrm{I}(t)}{t}\leq\limsup_{t\to 0}\int_\Omega g_t(x)\,\mathrm{d}x=\int_\Omega\int_{\overline{\Omega}^c-x}\limsup_{t\to 0}\mathbb{P}_{z+x}(X_t\in\Omega)\,\nu(\mathrm{d}z)\,\mathrm{d}x=0.$$ □

As a consequence of Theorem 3.2, one can prove the following result. Note that, unlike Theorem 3.3, we do not assume that $\mathrm{Per}(\Omega)<\infty$ in the result below.

Theorem 3.4. *Let $\mathbf{X}$ be a Lévy process of bounded variation with $\gamma_0=0$. If $\Omega\subset\mathbb{R}^d$ is an open set of finite Lebesgue measure, then we have*

$$\limsup_{t\to 0}\frac{|\Omega|-Q_\Omega(t)}{t}=\mathrm{Per}_{\mathbf{X}}(\Omega)+\limsup_{t\to 0}\frac{1}{t}\int_\Omega\mathbb{E}_x\left[\tau_\Omega<t,X_{\tau_\Omega}\in\partial\Omega,\mathbb{P}_{X_{\tau_\Omega}}\left(X_{t-\tau_\Omega}\in\Omega\right)\right]\mathrm{d}x$$

and

$$\liminf_{t\to 0}\frac{|\Omega|-Q_\Omega(t)}{t}=\mathrm{Per}_{\mathbf{X}}(\Omega)+\liminf_{t\to 0}\frac{1}{t}\int_\Omega\mathbb{E}_x\left[\tau_\Omega<t,X_{\tau_\Omega}\in\partial\Omega,\mathbb{P}_{X_{\tau_\Omega}}\left(X_{t-\tau_\Omega}\in\Omega\right)\right]\mathrm{d}x.$$

In particular if $\mathbb{P}^x\left(X_{\tau_\Omega}\in\partial\Omega\right)=0$ for almost every $x\in\Omega$ we have

$$\lim_{t\to 0}\frac{|\Omega|-Q_\Omega(t)}{t}=\mathrm{Per}_{\mathbf{X}}(\Omega).$$

Proof. If $\int_\Omega\nu(\Omega^c-y)\,\mathrm{d}y<\infty$, the proof is the same as the proof of Theorem 3.3 using Theorem 3.2 instead of [8, Theorem 3]. The case of $\int_\Omega\nu(\Omega^c-y)\,\mathrm{d}y=\infty$ is a consequence of Theorem 3.2 and the fact that $H_\Omega(t)\geq Q_\Omega(t)$. □

A domain $\Omega \subset \mathbb{R}^d$ is called a Lipschitz domain if there exist $R > 0$ and $\Lambda > 0$ such that, for every $z \in \partial\Omega$, there exist a function $\phi = \phi_x : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$ and $|\phi(x) - \phi(y)| \leq \Lambda|x - y|$, and an orthonormal coordinate system CS_z with origin at z such that

$$D \cap B(z, R) = B(z, R) \cap \left\{ y = (y_1, \dots, y_{d-1}, y_d) = \left(\tilde{y}, y_d\right) \text{ in } CS_z : y_d > \phi\left(\tilde{y}\right) \right\}.$$

Combining the above result with [25, Theorem 1], we immediately get the following

Corollary 3.5. *Suppose that* $\mathbf{X}$ *is an isotropic Lévy process of bounded variation and has an infinite Lévy measure. If* Ω *is a Lipschitz domain of finite Lebesgue measure, then*

$$\lim_{t \to 0} \frac{|\Omega| - Q_\Omega(t)}{t} = \mathrm{Per}_{\mathbf{X}}(\Omega).$$

Corollary 3.6. *Suppose that* $\mathbf{X}$ *is a Lévy process of bounded variation and there exist* $\alpha \in (0, 1)$ *and* $C > 0$ *such that*

$$\frac{C}{|\xi|^\alpha + 1} \leq \Re\left(\frac{1}{\psi(\xi) + 1}\right), \quad \xi \in \mathbb{R}^d, \tag{3.4}$$

where $\Re$ *represents the real part of its argument. If* Ω *is a Lipschitz domain of finite Lebesgue measure, then*

$$\lim_{t \to 0} \frac{|\Omega| - Q_\Omega(t)}{t} = \mathrm{Per}_{\mathbf{X}}(\Omega).$$

Proof. Since $\mathbf{X}$ is a process of bounded variation we have, for $|\xi| \geq 1$,

$$\Re\psi(\xi)/|\xi| \leq 2 \int_{\mathbb{R}^d} (1 \wedge |\xi z|)^2 \, \nu(dz)/|\xi| \leq 2 \int_{\mathbb{R}^d} \left(|\xi|^{-1} \wedge |z|\right) \nu(dz).$$

Moreover

$$|\Im(\psi(\xi) - i\langle\gamma_0, \xi\rangle)|/|\xi| \leq \int_{\mathbb{R}^d} (1 \wedge |\xi z|) \, \nu(dz)/|\xi| \leq \int_{\mathbb{R}^d} \left(|\xi|^{-1} \wedge |z|\right) \nu(dz),$$

where $\Im$ represents the imaginary part of its argument. Assume that $\gamma_0 \neq 0$. Since $\int_{\mathbb{R}^d} (1 \wedge |z|) \, \nu(dz) < \infty$, by the dominated convergence theorem we have

$$0 \leq \lim_{r \to \infty} r\Re\left(\frac{1}{\psi(\gamma_0 r) + 1}\right) \leq \lim_{r \to \infty} \frac{(1/r) + \Re\psi(\gamma_0 r)/r}{(|\gamma_0|^2 + (\Im(\psi(\gamma_0 r) - i|\gamma_0|^2 r)/r)^2} = 0,$$

which contradicts (3.4). Hence $\gamma_0 = 0$.

It is well known that the Hausdorff dimension of the boundary of a Lipschitz domain is $d - 1$ (see [11]). Therefore the claim is a consequence of Theorem 3.4 combined with [17, Theorem 3.3] and [26, Theorem 4 and Remark]. □

Combining Theorem 3.4 with [7] and [5], we get the following

Corollary 3.7. *Let* $d = 1$ *and let* Ω *be open. Assume that* $\mathbf{X}$ *is of bounded variation and* $\gamma_0 = 0$, *then*

$$\lim_{t \to 0} \frac{|\Omega| - Q_\Omega(t)}{t} = \mathrm{Per}_{\mathbf{X}}(\Omega).$$

Proof. We use here that $\{0\}$ is polar ([7, Theoreme 8] if the Lévy measure is infinite and [5, Theorem II.16] in case of a compound Poisson process) and therefore $\partial\Omega$ is polar. □

If there exists a nonzero drift γ_0, then the small time asymptotic behaviors of the heat content and the spectral heat content can be different. We illustrate this by the simple example below.

Example 3.8. Let $\gamma \in \mathbb{R} \setminus \{0\}$. We consider $\Omega = (-1, 0) \cup (0, 1)$ and a deterministic process $X_t = \gamma t$. Then $H(t) = (|\gamma|t) \wedge 2$ and $|\Omega| - Q_\Omega(t) = (2|\gamma|t) \wedge 2$. That is

$$2|\gamma| = \lim_{t \to 0} \frac{|\Omega| - Q_\Omega(t)}{t} \neq \lim_{t \to 0} \frac{|\Omega| - H_\Omega(t)}{t} = |\gamma|.$$

4 | PROCESSES OF UNBOUNDED VARIATION IN $\mathbb{R}$

In this section we study the small time asymptotic behavior of the spectral heat content for Lévy processes on the real line. For technical reasons, we will only deal with symmetric Lévy processes. We consider two different cases separately. In the first case, we assume that the characteristic exponent $\psi(\xi)$ of $\mathbf{X}$ is regularly varying of index α at infinity for some $\alpha \in (1,2]$. In the second case, we assume that $\mathbf{X}$ is a symmetric Lévy process whose characteristic exponent is $\psi(\xi) = |\xi|$, that is, $\mathbf{X}$ is a Cauchy process.

For any $\epsilon > 0$, let $\Omega_\epsilon := \{x \in \Omega : \mathrm{dist}(\{x\}, \partial\Omega) < \epsilon\}$.

Lemma 4.1. *For any $\varepsilon > 0$, we have*

$$\int_{\Omega\setminus\Omega_\varepsilon} \mathbb{P}_x\big(\tau_\Omega \le t\big)\mathrm{d}x \le C_1\, |\Omega \setminus \Omega_\varepsilon| t h(\varepsilon).$$

Proof. Note that, for any $x \in \Omega \setminus \Omega_\varepsilon$, $\mathbb{P}_x\big(\tau_\Omega \le t\big) \le \mathbb{P}_x\big(\sup_{s\le t}|X_s - x| \ge \varepsilon\big) = \mathbb{P}\big(\sup_{s\le t} X_s \ge \varepsilon\big)$. Thus it follows from (2.1) that

$$\int_{\Omega\setminus\Omega_\varepsilon} \mathbb{P}_x\big(\tau_\Omega \le t\big)\mathrm{d}x \le C_1\, |\Omega \setminus \Omega_\varepsilon| t h(\varepsilon).$$

□

Every open set Ω in $\mathbb{R}$ can be written as the union of countably many disjoint open intervals: $\Omega = \bigcup_i (a_i, b_i)$. Let $\Omega = \bigcup_i (a_i, b_i)$ and $\partial^{ad}\Omega := \big\{b_j : \text{ there exists } i \neq j \text{ such that } b_j = a_i\big\}$ be the subset of $\partial\Omega$ which consists of common boundary points of adjacent components of Ω. Let

$$\tilde{\Omega} := \Omega \cup \partial^{ad}\Omega \tag{4.1}$$

be the augmented set of Ω. Note that the distance between any two distinct components of $\tilde{\Omega}$ is always strictly positive.

Recall that

$$f_\Omega(y) = |\Omega| - |\Omega \cap (\Omega + y)| = \int_{\mathbb{R}} 1_\Omega(x)\,\mathrm{d}x - \int_{\mathbb{R}} 1_\Omega(x)1_\Omega(x-y)\,\mathrm{d}x = \int_{\mathbb{R}} 1_\Omega(x)1_{\Omega^c}(x-y)\,\mathrm{d}x.$$

For any Lévy process $\mathbf{X}$ and $t \ge 0$, we define $\overline{X}_t := \sup_{s\in[0,t]} X_s$ and $\underline{X}_t := \inf_{s\in[0,t]} X_s$.

Let $\psi^*(u) = \sup_{\xi\in[0,u]} \psi(\xi)$, $u \ge 0$, and let $\psi^{-1}(u) = \inf\{s \ge 0 : \psi^*(s) \ge u\}$ be the generalized inverse of ψ^*.

4.1 | $\psi \in \mathcal{R}_\alpha$, $\alpha \in (1,2]$

In this subsection we study the small time asymptotic behavior of the spectral heat content of general open sets of finite Lebesgue measure with respect to symmetric Lévy processes in $\mathbb{R}$. We assume that $\mathbf{X}$ is a symmetric Lévy process with characteristic exponent ψ and that there exists $\alpha \in (1,2]$ such that $\psi \in \mathcal{R}_\alpha$. Here are a few examples of characteristic exponents belonging to $\mathcal{R}_\alpha$: (1) $\psi(\lambda) = \lambda^\alpha$, $\alpha \in (0,2]$; (2) $\psi(\lambda) = \lambda^\alpha + \lambda^\beta$, $0 < \beta < \alpha \le 2$; (3) $\psi(\lambda) = \big(\lambda^2+1\big)^{\alpha/2} - 1$; (4) $\psi(\lambda) = \lambda^\alpha\big(\log\big(\lambda^2+1\big)\big)^{\beta/2}$, $\alpha \in (0,2)$ and $0 < \beta < 2-\alpha$; (5) $\psi(\lambda) = \lambda^\alpha\big(\log\big(\lambda^2+1\big)\big)^{-\beta/2}$, $0 < \beta < \alpha < 2$.

For $y \in \mathbb{R}$, let $T_y = \inf\{t > 0 : X_t = y\}$ be the first time the process $\mathbf{X}$ hits y and $T_y^{(\alpha)}$ the first time the symmetric α-stable process $\mathbf{S}^{(\alpha)}$ (with characteristic exponent $|\xi|^\alpha$) hits y.

Here is the main result of this section:

Theorem 4.2. *Suppose that $\mathbf{X}$ is a symmetric Lévy process with characteristic exponent $\psi \in \mathcal{R}_\alpha$ for some $\alpha \in (1,2]$. Let Ω be an open set in $\mathbb{R}$ with $|\Omega| < \infty$. Let A be the number of components of $\tilde{\Omega}$ and let B be number of points in $\partial^{ad}\Omega$. Then we have*

$$\lim_{t\to 0} \psi^{-1}(1/t)\big(|\Omega| - Q_\Omega(t)\big) = 2A\mathbb{E}\Big[\overline{S^{(\alpha)}}_1\Big] + 2BC_1, \tag{4.2}$$

where $C_1 = \int_0^\infty \mathbb{P}\big(T_u^{(\alpha)} \le 1\big)\,\mathrm{d}u < \mathbb{E}\big[\overline{S^{(\alpha)}}_1\big] < \infty$.

In the case of isotropic α-stable processes, we have $\psi^{-1}(1/t) = t^{-1/\alpha}$.

Remark 4.3. Note that A can be finite even if the number of components in Ω is infinite. For example, the open set $\Omega = \bigcup_{n=1}^{\infty}\left(\frac{1}{n+1},\frac{1}{n}\right)$ has infinitely many components, but $A = 1$. Also if Ω has infinitely many components, either A or B must be infinite and therefore we have $\lim_{t\to 0}\psi^{-1}(1/t)\big(|\Omega| - Q_\Omega(t)\big) = \infty$.

Under the assumptions of this subsection, $e^{-t\psi(\xi)} \in L^1\big(\mathbb{R}^1\big)$ and thus, by [24, Proposition 28.1], the process $\mathbf{X}$ has a transition density, and thus $H_\Omega(t) = H_{\tilde{\Omega}}(t)$. Hence it follows from [8, Theorem 2] that when $\Omega \subset \mathbb{R}$ has infinitely many components but $\tilde{\Omega}$ has only finitely many components, we have

$$\lim_{t\to 0}\psi^{-1}(1/t)\big(|\Omega| - H_\Omega(t)\big) = \frac{\Gamma\left(1-\frac{1}{\alpha}\right)}{\pi}\mathrm{Per}(\Omega) = \frac{2\Gamma\left(1-\frac{1}{\alpha}\right)}{\pi}\mathrm{A} < \infty.$$

Remark 4.4. In the case of Brownian motion, we have from [3, Equation 4.4]

$$C_1 = \int_0^\infty \mathbb{P}\left(T_u^{\mathbf{B}} \le 1\right) \mathrm{d}u = \int_0^\infty \mathbb{P}\big(\overline{B}_1 \ge u\big)\mathrm{d}u = \mathbb{E}\big[\overline{B}_1\big] = \frac{2}{\sqrt{\pi}}.$$

Note that, by the definitions, $A + B$ is equal to the number of components in Ω. Hence in this case (4.2) becomes

$$\lim_{t\to 0}\frac{|\Omega| - Q_\Omega^{\mathbf{B}}(t)}{\sqrt{t}} = 2 \times (\text{the number of components in } \Omega) \times \frac{2}{\sqrt{\pi}}.$$

We now give some preliminary results to prepare for the proof of Theorem 4.2. It is easy to see that, under the assumptions of Theorem 4.2, $\mathbf{X}$ is of unbounded variation. Since $\mathbf{X}$ is symmetric, it follows from [16, Corollary 1] that

$$\frac{1}{2}\psi^*\left(r^{-1}\right) \le h(r) \le 24\psi^*\left(r^{-1}\right). \tag{4.3}$$

Also it follows from [6, Theorem 1.5.3] that, for each $R > 0$, there exists a constant $c = c(R) > 0$ such that

$$\psi^*(r) \le c\psi(r), \quad \text{for } r \ge R. \tag{4.4}$$

Since $\psi \in \mathcal{R}_\alpha$ for some $\alpha \in (1,2]$ and $\mathbf{X}$ is symmetric, we have from [6, Proposition 1.3.6-(v)] $\psi(\xi) = |\xi|^\alpha \ell(\xi) \ge |\xi|^{\frac{\alpha+1}{2}}$ when $|\xi| \to \infty$ and ℓ is slowly varying at ∞. Hence this gives

$$\int_{\mathbb{R}} \frac{1}{1+\psi(\xi)}\,\mathrm{d}\xi < \infty.$$

It follows from [7, Theorem 8] that $\big\{y \in \mathbb{R} : \mathbb{P}\big(T_y < \infty\big) > 0\big\} = \mathbb{R}$.

Lemma 4.5. *Suppose that $\mathbf{X}$ is a symmetric Lévy process with characteristic exponent $\psi \in \mathcal{R}_\alpha$ for some $\alpha \in (1,2]$. There exists $\varepsilon_1 > 0$ such that for any $\varepsilon \le \varepsilon_1$,*

$$\lim_{t\to 0}\psi^{-1}(1/t)\int_0^\varepsilon \mathbb{P}\left(T_y \le t\right)\mathrm{d}y = \int_0^\infty \mathbb{P}\left(T_u^{(\alpha)} \le 1\right)\mathrm{d}u.$$

Proof. Define a process $\mathbf{Y}^{(t)} = \big(Y_s^{(t)}\big)_{s\ge 0}$ by $Y_s^{(t)} = \psi^{-1}(1/t)X_{ts}$. The characteristic exponent of $\mathbf{Y}^{(t)}$ is $\psi^{(t)}(\xi) = t\psi\big(\psi^{-1}(1/t)\xi\big)$. Note that

$$\lim_{t\to 0}\psi^{(t)}(\xi) = \lim_{t\to 0}\frac{\psi\left(\psi^{-1}(1/t)\xi\right)}{\psi\left(\psi^{-1}(1/t)\right)} = |\xi|^\alpha.$$

Observe that by the change of variables $u = \psi^{-1}(1/t)y$,

$$\psi^{-1}(1/t)\int_0^\varepsilon \mathbb{P}\left(T_y \le t\right)\mathrm{d}y = \psi^{-1}(1/t)\int_0^\varepsilon \mathbb{P}\left(T^{\mathbf{Y}^{(t)}}_{\psi^{-1}(1/t)y} \le 1\right)\mathrm{d}y = \int_0^{\varepsilon\psi^{-1}(1/t)} \mathbb{P}\left(T_u^{\mathbf{Y}^{(t)}} \le 1\right)\mathrm{d}u.$$

Fix $0<\delta<\alpha-1$. Since $\psi\in\mathcal{R}_\alpha$, it follows from [6, Theorem 1.5.6] that there exists $x_0>0$ such that

$$\frac{\psi\big(\psi^{-1}(1/t)1/u\big)}{\psi\big(\psi^{-1}(1/t)\big)}\le\frac{2}{u^{\alpha-\delta}}\tag{4.5}$$

for all $\psi^{-1}(1/t)\ge x_0$ and $1\le u\le\frac{\psi^{-1}(1/t)}{x_0}$. Hence by [5, Theorem II.19.(iii)] and the dominated convergence theorem,

$$\lim_{t\to0}\mathbb{P}\Big(T_u^{\mathbf{Y}^{(t)}}\le1\Big)=\mathbb{P}\big(T_u^{(\alpha)}\le1\big).\tag{4.6}$$

Let $\varepsilon_1:=\frac{1}{x_0}$. It follows from (4.4) that there exists $c=c(x_0)>0$ such that

$$\psi^*(r)\le c(x_0)\psi(r),\quad r\ge x_0.\tag{4.7}$$

Take $M>0$. It follows from the dominated convergence theorem and (4.6) that

$$\lim_{t\to0}\int_0^M\mathbb{P}\Big(T_u^{\mathbf{Y}^{(t)}}\le1\Big)\,\mathrm{d}u=\int_0^M\mathbb{P}\big(T_u^{(\alpha)}\le1\big)\,\mathrm{d}u.\tag{4.8}$$

Suppose $\varepsilon\le\varepsilon_1$ and $M\le u\le\varepsilon\psi^{-1}(1/t)$. Then from (2.1), (4.3), (4.5) and (4.7) we get that

$$\begin{aligned}\mathbb{P}\Big(T_u^{\mathbf{Y}^{(t)}}\le1\Big)&\le\mathbb{P}\Big(\overline{Y^{(t)}}_1\ge u\Big)\le C_1h^{\mathbf{Y}^{(t)}}(u)\le24C_1t\psi^*\big(\psi^{-1}(1/t)1/u\big)\\&\le24C_1c(x_0)t\psi\big(\psi^{-1}(1/t)1/u\big)=24C_1c(x_0)\frac{\psi\big(\psi^{-1}(1/t)1/u\big)}{\psi\big(\psi^{-1}(1/t)\big)}\le\frac{48C_1c(x_0)}{u^{\alpha-\delta}}.\end{aligned}$$

Hence we have

$$\int_M^{\varepsilon\psi^{-1}(1/t)}\mathbb{P}\Big(T_u^{\mathbf{Y}^{(t)}}\le1\Big)\,\mathrm{d}u\le\int_M^{\varepsilon\psi^{-1}(1/t)}\frac{c(d,x_0)}{u^{\alpha-\delta}}\,\mathrm{d}u\le\int_M^{\infty}\frac{c(d,x_0)}{u^{\alpha-\delta}}\,\mathrm{d}u=\frac{c(d,x_0)}{\alpha-\delta-1}M^{-\alpha+\delta+1}\tag{4.9}$$

for all $\varepsilon\le\varepsilon_1$, $M<\varepsilon\psi^{-1}(1/t)$, and $\psi^{-1}(1/t)\ge x_0$. By letting $t\to0$ and then letting $M\to\infty$ in (4.8) and (4.9), we reach the conclusion of the lemma. □

Lemma 4.6. *Suppose that* $\mathbf{X}$ *is a symmetric Lévy process with characteristic exponent* $\psi\in\mathcal{R}_\alpha$ *for some* $\alpha\in(1,2]$. *There exists* $\varepsilon_2>0$ *such that for all* $\varepsilon\le\varepsilon_2$,

$$\lim_{t\to0}\psi^{-1}(1/t)\int_0^\varepsilon\mathbb{P}\Big(\overline{X}_t\ge x\Big)\,\mathrm{d}x=\mathbb{E}\Big[\overline{S^{(\alpha)}}_1\Big].$$

Proof. Let $\mathbf{Y}^{(t)}$ be the process defined in the proof of Lemma 4.5. Recall from the proof of Lemma 4.5 that the characteristic exponent $\psi^{(t)}$ of $\mathbf{Y}^{(t)}$ is

$$\lim_{t\to0}\psi^{(t)}(\xi)=|\xi|^\alpha.\tag{4.10}$$

Moreover,

$$\psi^{-1}(1/t)\int_0^\varepsilon\mathbb{P}\Big(\overline{X}_t\ge x\Big)\,\mathrm{d}x=\psi^{-1}(1/t)\int_0^\varepsilon\mathbb{P}\Big(\overline{Y^{(t)}}_1\ge\psi^{-1}(1/t)x\Big)\,\mathrm{d}x=\int_0^{\varepsilon\psi^{-1}(1/t)}\mathbb{P}\Big(\overline{Y_1^{(t)}}\ge x\Big)\,\mathrm{d}x.$$

Using (2.1), (4.10), and [15, Theorem VI.5.5], we can get that $\overline{Y^{(t)}}_1\xrightarrow{D}\overline{S^{(\alpha)}}_1$ (since $x\mapsto\sup_{t\in[0,1]}x(t)$ is a continuous functional on the Skorohod space). The rest of the proof is identical to the proof of Lemma 4.5. □

Lemma 4.7. *Suppose that* $\mathbf{X}$ *is a symmetric Lévy process with characteristic exponent* $\psi\in\mathcal{R}_\alpha$ *for some* $\alpha\in(1,2]$. *Let* $\Omega=\bigcup_i(a_i,b_i)$ *with* $|\Omega|=\sum_i(b_i-a_i)<\infty$. *Suppose that* $a_i\notin\partial^{ad}\Omega$ *and* $\varepsilon<\frac12((b_i-a_i)\wedge\varepsilon_2)$. *Then we have*

$$\lim_{t\to0}\psi^{-1}(1/t)\int_{a_i}^{a_i+\varepsilon}\mathbb{P}_x\big(\tau_\Omega\le t\big)\mathrm{d}x=\mathbb{E}\Big[\overline{S^{(\alpha)}}_1\Big].$$

Similarly, if $b_i \notin \partial^{ad}\Omega$ and $\varepsilon < \frac{1}{2}\big((b_i - a_i) \wedge \varepsilon_2\big)$, then

$$\lim_{t\to 0} \psi^{-1}(1/t) \int_{b_i-\varepsilon}^{b_i} \mathbb{P}_x\big(\tau_\Omega \le t\big)\mathrm{d}x = \mathbb{E}\Big[\overline{S^{(\alpha)}}_1\Big].$$

Proof. By the symmetry of $\mathbf{X}$ it is enough to prove the first limit. Suppose that $a_i \notin \partial^{ad}\Omega$, $\varepsilon < \frac{b_i - a_i}{2}$ and $x \in (a_i, a_i + \varepsilon)$. Note that, under $\mathbb{P}_x$, depending on whether $\overline{X}_t > a_i + 2\varepsilon$ or $\underline{X}_t < a_i$, the event $\big\{\tau_\Omega > t\big\}$ can be written as

$$\{\tau_\Omega > t\} = \Big\{a_i < \underline{X}_t \le \overline{X}_t < a_i + 2\varepsilon\Big\} \cup \big\{\tau_\Omega > t, \underline{X}_t < a_i\big\} \cup \Big\{\tau_\Omega > t, \overline{X}_t \ge a_i + 2\varepsilon\Big\}.$$

Note that the first event of the display above is disjoint with the union of the last two events. Hence we have

$$\mathbb{P}_x\big(\tau_\Omega > t\big) = \mathbb{P}_x\Big(a_i < \underline{X}_t \le \overline{X}_t < a_i + 2\varepsilon\Big) + \mathbb{P}_x\Big(\big\{\tau_\Omega > t, \underline{X}_t < a_i\big\} \cup \Big\{\tau_\Omega > t, \overline{X}_t \ge a_i + 2\varepsilon\Big\}\Big).$$

This implies that

$$\begin{aligned}
\mathbb{P}_x\big(\tau_\Omega \le t\big) &= 1 - \mathbb{P}_x\Big(a_i < \underline{X}_t \le \overline{X}_t < a_i + 2\varepsilon\Big) - \mathbb{P}_x\Big(\big\{\tau_\Omega > t, \underline{X}_t \le a_i\big\} \cup \Big\{\tau_\Omega > t, \overline{X}_t \ge a_i + 2\varepsilon\Big\}\Big)\\
&= \mathbb{P}_x\Big(\big\{\underline{X}_t \le a_i\big\} \cup \Big\{\overline{X}_t \ge a_i + 2\varepsilon\Big\}\Big) - \mathbb{P}_x\Big(\big\{\tau_\Omega > t, \underline{X}_t \le a_i\big\} \cup \Big\{\tau_\Omega > t, \overline{X}_t \ge a_i + 2\varepsilon\Big\}\Big)\\
&= \mathbb{P}_x\big(\underline{X}_t \le a_i\big) + \mathbb{P}_x\Big(\overline{X}_t \ge a_i + 2\varepsilon\Big) - \mathbb{P}_x\Big(\overline{X}_t \ge a_i + 2\varepsilon \text{ and } \underline{X}_t \le a_i\Big)\\
&\quad - \mathbb{P}_x\big(\tau_\Omega > t, \underline{X}_t \le a_i\big) - \mathbb{P}_x\Big(\tau_\Omega > t, \overline{X}_t \ge a_i + 2\varepsilon\Big) + \mathbb{P}_x\Big(\tau_\Omega > t, \underline{X}_t \le a_i \text{ and } \overline{X}_t \ge a_i + 2\varepsilon\Big). \qquad (4.11)
\end{aligned}$$

Let $b := \sup\{x \in \Omega : x < a_i\}$. Since $a_i \notin \partial^{ad}\Omega$, we have either $\{x \in \Omega : x < a_i\} = \emptyset$ thus $b = -\infty$ or $b < a_i$. Hence we have either

$$\big\{\tau_\Omega > t, \underline{X}_t \le a_i\big\} = \emptyset$$

or

$$\big\{\tau_\Omega > t, \underline{X}_t \le a_i\big\} = \big\{\tau_\Omega > t, \underline{X}_t \le b\big\}. \qquad (4.12)$$

We will deal with the second case since the first case is similar and much easier. It follows from (4.12) that (4.11) can be written as

$$\begin{aligned}
\mathbb{P}_x\big(\tau_\Omega \le t\big) &= \mathbb{P}_x\big(\underline{X}_t \le a_i\big) + \mathbb{P}_x\Big(\overline{X}_t \ge a_i + 2\varepsilon\Big) - \mathbb{P}_x\Big(\overline{X}_t \ge a_i + 2\varepsilon \text{ and } \underline{X}_t \le a_i\Big)\\
&\quad - \mathbb{P}_x\big(\tau_\Omega > t, \underline{X}_t \le b\big) - \mathbb{P}_x\Big(\tau_\Omega > t, \overline{X}_t \ge a_i + 2\varepsilon\Big) + \mathbb{P}_x\Big(\tau_\Omega > t, \underline{X}_t \le a_i \text{ and } \overline{X}_t \ge a_i + 2\varepsilon\Big).
\end{aligned}$$

Hence

$$\mathbb{P}_x\big(\underline{X}_t \le a_i\big) - 2\mathbb{P}_x\Big(\overline{X}_t \ge a_i + 2\varepsilon\Big) - \mathbb{P}_x\big(\underline{X}_t \le b\big) \le \mathbb{P}_x\big(\tau_\Omega \le t\big) \le \mathbb{P}_x\big(\underline{X}_t \le a_i\big) + 2\mathbb{P}_x\Big(\overline{X}_t \ge a_i + 2\varepsilon\Big).$$

Note that by the symmetry of $\mathbf{X}$ we have

$$\int_{a_i}^{a_i+\varepsilon} \mathbb{P}_x\big(\underline{X}_t \le a_i\big)\mathrm{d}x = \int_0^{\varepsilon} \mathbb{P}\Big(\overline{X}_t \ge y\Big)\,\mathrm{d}y.$$

Hence from Lemma 4.6,

$$\lim_{t\to 0} \psi^{-1}(1/t) \int_{a_i}^{a_i+\varepsilon} \mathbb{P}_x\big(\underline{X}_t \le a_i\big)\mathrm{d}x = \mathbb{E}\Big[\overline{S^{(\alpha)}}_1\Big].$$

By (2.1) we have

$$\int_{a_i}^{a_i+\varepsilon} \mathbb{P}_x\left(\overline{X}_t \geq a_i + 2\varepsilon\right) \mathrm{d}x = \int_{\varepsilon}^{2\varepsilon} \mathbb{P}\left(\overline{X}_t \geq y\right) \mathrm{d}y \leq C_1 t \varepsilon h(\varepsilon).$$

Since $\psi^{-1} \in \mathcal{R}_{1/\alpha}$ (see [6, Theorems 1.5.3 and 1.5.12]), we have

$$\lim_{t\to 0} \psi^{-1}(1/t) \int_{a_i}^{a_i+\varepsilon} \mathbb{P}_x\left(\overline{X}_t \geq a_i + 2\varepsilon\right) \mathrm{d}x = 0.$$

Since $a_i - b > 0$, by the symmetry of $\mathbf{X}$ and the same argument as above we get

$$\lim_{t\to 0} \psi^{-1}(1/t) \int_{a_i}^{a_i+\varepsilon} \mathbb{P}_x\left(\underline{X}_t \leq b\right) \mathrm{d}x = 0.$$

The proof is now complete. □

Lemma 4.8. *Suppose that* $\mathbf{X}$ *is a symmetric Lévy process with characteristic exponent* $\psi \in \mathcal{R}_\alpha$ *for some* $\alpha \in (1, 2]$. *Let* $\Omega = \bigcup_i (a_i, b_i)$ *with* $|\Omega| = \sum_i (b_i - a_i) < \infty$. *If* $a_i \in \partial^{ad}\Omega$ *and* $\varepsilon < \frac{1}{2}\left((b_i - a_i) \wedge \varepsilon_1\right)$, *then*

$$\lim_{t\to 0} \psi^{-1}(1/t) \int_{a_i}^{a_i+\varepsilon} \mathbb{P}_x\left(\tau_\Omega \leq t\right) \mathrm{d}x = \int_0^\infty \mathbb{P}\left(T_u^{(\alpha)} \leq 1\right) \mathrm{d}u.$$

Similarly, if $b_i \in \partial^{ad}\Omega$ *and* $\varepsilon < \frac{1}{2}\left((b_i - a_i) \wedge \varepsilon_1\right)$, *then*

$$\lim_{t\to 0} \psi^{-1}(1/t) \int_{b_i-\varepsilon}^{b_i} \mathbb{P}_x\left(\tau_\Omega \leq t\right) \mathrm{d}x = \int_0^\infty \mathbb{P}\left(T_u^{(\alpha)} \leq 1\right) \mathrm{d}u.$$

Proof. Suppose that $a_i \in \partial^{ad}\Omega$, $\varepsilon < \frac{b_i - a_i}{2}$ and $x \in (a_i, a_i + \varepsilon)$. Let $\left(a_j, b_j\right)$ with $b_j = a_i$ be the component of Ω which is adjacent to (a_i, b_i). Then we have under $\mathbb{P}_x$,

$$\{\tau_\Omega \leq t\} = \{T_{a_i} \leq t\} \cup \left\{\tau_\Omega \leq t, T_{a_i} > t\right\} \subset \{T_{a_i} \leq t\} \cup \left\{\overline{X}_t \geq a_i + 2\varepsilon \text{ or } \underline{X}_t \leq a_j\right\}.$$

It follows from an argument similar to that in the proof of Lemma 4.7 we have

$$\lim_{t\to 0} \psi^{-1}(1/t) \int_{a_i}^{a_i+\varepsilon} \mathbb{P}_x\left(\overline{X}_t \geq a_i + 2\varepsilon \text{ or } \underline{X}_t \leq a_j\right) \mathrm{d}x = 0.$$

Hence we have

$$\limsup_{t\to 0} \psi^{-1}(1/t) \int_{a_i}^{a_i+\varepsilon} \mathbb{P}_x\left(\tau_\Omega \leq t\right) \mathrm{d}x \leq \limsup_{t\to 0} \psi^{-1}(1/t) \int_{a_i}^{a_i+\varepsilon} \mathbb{P}_x\left(T_{a_i} \leq t\right) \mathrm{d}x$$

and

$$\liminf_{t\to 0} \psi^{-1}(1/t) \int_{a_i}^{a_i+\varepsilon} \mathbb{P}_x\left(\tau_\Omega \leq t\right) \mathrm{d}x \geq \liminf_{t\to 0} \psi^{-1}(1/t) \int_{a_i}^{a_i+\varepsilon} \mathbb{P}_x\left(T_{a_i} \leq t\right) \mathrm{d}x.$$

Now using Lemma 4.5 we obtain the claim. □

Now we state a result handling the case when Ω has infinitely many components.

Lemma 4.9. *Suppose that* $\mathbf{X}$ *is a symmetric Lévy process with characteristic exponent* $\psi \in \mathcal{R}_\alpha$ *for some* $\alpha \in (1, 2]$. *If* $\Omega \subset \mathbb{R}$ *is of finite Lebesgue measure and has infinitely many components, then*

$$\liminf_{t\to 0} \psi^{-1}(1/t)\left(|\Omega| - Q_\Omega(t)\right) = \infty.$$

Proof. If Ω has infinitely many components, either A, the number of components in $\tilde{\Omega}$, or B, the number of points in $\partial^{ad}\Omega$, is infinite. Suppose that $A = \infty$. Let $\Omega = \bigcup_{i=1}^\infty (a_i, b_i)$. Then there must be infinitely many i such that $a_i \notin \partial^{ad}\Omega$ or $b_i \notin \partial^{ad}\Omega$. Let

$\mathcal{I} = \{i : a_i \notin \partial^{ad}\Omega \text{ or } b_i \notin \partial^{ad}\Omega\}$. Given N, take $\varepsilon = \varepsilon(N)$ small so that there are at least N many i's with $\varepsilon < \frac{1}{2}((b_i - a_i) \wedge \varepsilon_1)$. Then it follows from Lemma 4.7 we have

$$\begin{aligned}\liminf_{t\to 0} \psi^{-1}(1/t)\big(|\Omega| - Q_\Omega(t)\big) &\geq \liminf_{t\to 0}\left(\sum_{i\in\mathcal{I}} \psi^{-1}(1/t)\left(\int_{a_i}^{a_i+\varepsilon} \mathbb{P}_x\big(\tau_\Omega \leq t\big)\mathrm{d}x + \int_{b_i-\varepsilon}^{b_i} \mathbb{P}_x\big(\tau_\Omega \leq t\big)\mathrm{d}x\right)\right) \\ &\geq N\mathbb{E}\left[\overline{S^{(\alpha)}}_1\right].\end{aligned}$$

Now the assertion follows by letting $N \to \infty$.

The case when $B = \infty$ can be proved in a similar way using Lemma 4.8. □

Now we show that $\int_0^\infty \mathbb{P}(T_u^{(\alpha)} \leq 1)\, du < \mathbb{E}\big[\overline{S^{(\alpha)}}_1\big]$.

Lemma 4.10. *Suppose $\alpha \in (1,2)$. Then we have $\int_0^\infty \mathbb{P}(T_u^{(\alpha)} \leq 1)\,\mathrm{d}u < \mathbb{E}\big[\overline{S^{(\alpha)}}_1\big]$.*

Proof. Since $\overline{S^{(\alpha)}}_1$ is nonnegative, its expectation can be written as $\mathbb{E}\big[\overline{S^{(\alpha)}}_1\big] = \int_0^\infty \mathbb{P}\Big(\overline{S^{(\alpha)}}_1 \geq u\Big)\mathrm{d}u$. Note that we have $\{T_u^{(\alpha)} \leq 1\} \subset \Big\{\overline{S^{(\alpha)}}_1 \geq u\Big\}$. Hence we have

$$\int_0^\infty \mathbb{P}\big(T_u^{(\alpha)} \leq 1\big)\, du \leq \int_0^\infty \mathbb{P}\Big(\overline{S^{(\alpha)}}_1 \geq u\Big)\,\mathrm{d}u = \mathbb{E}\left[\overline{S^{(\alpha)}}_1\right].$$

Assume by contrary that $\int_0^\infty \mathbb{P}(T_u^{(\alpha)} \leq 1)\,\mathrm{d}u = \int_0^\infty \mathbb{P}\Big(\overline{S^{(\alpha)}}_1 \geq u\Big)\mathrm{d}u$. Then we must have $\mathbb{P}(T_u^{(\alpha)} \leq 1) = \mathbb{P}\Big(\overline{S^{(\alpha)}}_1 \geq u\Big)$ for a.s. $u \in (0,\infty)$.

It follows from [5, Proposition VIII-4] we have

$$\mathbb{P}\Big(\overline{S^{(\alpha)}}_1 \geq u\Big) \sim ku^{-\alpha} \text{ as } u \to \infty \tag{4.13}$$

for some constant k. On the other hand, it follows from [34, Theorem 5.3]

$$\mathbb{P}\Big(T_1^{(\alpha)} < t\Big) = c_1 \int_0^t (t-s)^{\frac{1}{\alpha}-1} p^{(\alpha)}(s,1)\,\mathrm{d}s,$$

where $p^{(\alpha)}(s,\cdot)$ is the transition density of the symmetric α-stable processes. Since $p^{(\alpha)}(s,1) \asymp s^{-\frac{1}{\alpha}} \wedge s$, we have for $t < 1$,

$$\int_0^t (t-s)^{\frac{1}{\alpha}-1} p^{(\alpha)}(s,1)\,\mathrm{d}s \leq c_2 \int_0^t t^{\frac{1}{\alpha}-1} s\,\mathrm{d}s \leq c_3 t^{\frac{1}{\alpha}+1}$$

for some constant $c_3 > 0$. By the scaling property $T_u^{(\alpha)}$ and $|u|^\alpha T_1^{(\alpha)}$ are equal in distribution. Hence we have

$$\mathbb{P}\big(T_u^{(\alpha)} \leq 1\big) = \mathbb{P}\left(T_1^{(\alpha)} \leq \frac{1}{u^\alpha}\right) \leq c_4 u^{-1-\alpha} \text{ for } u \geq 1. \tag{4.14}$$

(4.13) and (4.14) yield a contradiction and we reach the conclusion of the lemma. □

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. If Ω has infinitely many components, the result follows from Lemma 4.9. Now assume that Ω has finitely many components. Write $\Omega = \bigcup_{i=1}^N (a_i, b_i)$ and let $\varepsilon = \frac{1}{2}\min_{1\leq i\leq N}\big((b_i - a_i) \wedge \varepsilon_1 \wedge \varepsilon_2\big)$, where ε_1 and ε_2 are constants in Lemmas 4.5 and 4.6, respectively. Let $\mathcal{B}$ be the set of points which are the common boundary points of two adjacent components of Ω and $\mathcal{A} = \bigcup_{i=1}^N \{a_i, b_i\} \setminus \mathcal{B}$. Then $|\mathcal{A}| = 2A$, $|\mathcal{B}| = B$, and $A + B = N$. It follows from Lemmas 4.1, 4.7 and 4.8 that

$$\begin{aligned}\lim_{t\to 0} \psi^{-1}(1/t)\big(|\Omega| - Q_\Omega(t)\big) &= \lim_{t\to 0} \psi^{-1}(1/t)\left(\int_{\Omega\setminus\Omega_\varepsilon} \mathbb{P}_x\big(\tau_\Omega \leq t\big)\mathrm{d}x\right) \\ &\quad + \lim_{t\to 0} \psi^{-1}(1/t) \sum_{a_i, b_k \in \mathcal{A}} \left(\int_{a_i}^{a_i+\varepsilon} \mathbb{P}_x\big(\tau_\Omega \leq t\big)\mathrm{d}x + \int_{b_k-\varepsilon}^{b_k} \mathbb{P}_x\big(\tau_\Omega \leq t\big)\mathrm{d}x\right)\end{aligned}$$

$$+\lim_{t\to 0}\psi^{-1}(1/t)\sum_{a_j,b_{j-1}\in\mathcal{B}}\left(\int_{a_j}^{a_j+\varepsilon}\mathbb{P}_x\big(\tau_\Omega\le t\big)\mathrm{d}x+\int_{b_{j-1}-\varepsilon}^{b_{j-1}}\mathbb{P}_x\big(\tau_\Omega\le t\big)\mathrm{d}x\right)$$

$$=2A\mathbb{E}\left[\overline{S^{(\alpha)}}_1\right]+2BC_1.$$

Finally Lemma 4.10 shows that $0<C_1<\mathbb{E}\left[\overline{S^{(\alpha)}}_1\right]<\infty$. □

4.2 | Cauchy process

First we give some preliminary results to prepare for the proof of Theorem 4.14. In this subsection we assume that $\mathbf{X}$ is a Cauchy process, that is, a symmetric 1-stable Lévy process, in $\mathbb{R}$ with characteristic exponent $\psi(\xi)=|\xi|$.

Lemma 4.11. *Suppose that* $\mathbf{X}$ *is a Cauchy process. Let* $\Omega=\bigcup_i(a_i,b_i)$ *with* $|\Omega|=\sum_i(b_i-a_i)<\infty$. *If* $a_i\notin\partial^{ad}\Omega$ *and* $\varepsilon<\frac{b_i-a_i}{2}$, *then*

$$\lim_{t\to 0}\frac{\int_{a_i}^{a_i+\varepsilon}\mathbb{P}_x\big(\tau_\Omega\le t\big)\mathrm{d}x}{t\ln(1/t)}=\frac{1}{\pi}.$$

Similarly, if $b_i\notin\partial^{ad}\Omega$ *and* $\varepsilon<\frac{b_i-a_i}{2}$, *then*

$$\lim_{t\to 0}\frac{\int_{b_i-\varepsilon}^{b_i}\mathbb{P}_x\big(\tau_\Omega\le t\big)\mathrm{d}x}{t\ln(1/t)}=\frac{1}{\pi}.$$

Proof. The proof is almost identical to the proof of Lemma 4.7 using [3, Proposition 4.3 (i)] instead of Lemma 4.6, so we omit the details. □

Now we address the issue when Ω has adjacent components. Recall the definition of augmented set $\tilde{\Omega}$ in (4.1). It is well known that, when $0<\alpha\le 1$, a single point is polar for the process hence $T_x=\inf\{s\,:\,X_s=x\}$ is almost surely infinite. Hence we have the following result.

Lemma 4.12. *If* $\mathbf{X}$ *is a Cauchy process, then* $Q_\Omega(t)=Q_{\tilde{\Omega}}(t)$.

Proof. By ([7, Theoreme 8]) $\{0\}$ is polar and therefore $\partial\Omega$ is polar as well. Hence

$$\mathbb{P}_x\big(\tau_{\tilde{\Omega}}>t\big)=\mathbb{P}_x\big(\tau_\Omega>t\big)$$

almost surely. This implies the claim. □

Lemma 4.13. *Suppose that* $\mathbf{X}$ *is a Cauchy process. If* Ω *is of finite Lebesgue measure and* $\tilde{\Omega}=\Omega\cup\partial^{ad}\Omega$ *has infinitely many components, then*

$$\liminf_{t\to 0}\frac{|\Omega|-Q_\Omega(t)}{t\ln(1/t)}=\infty.$$

Proof. The proof is very similar to the proof of Lemma 4.9 using Lemma 4.11. □

Theorem 4.14. *Let* $\mathbf{X}$ *be a Cauchy process. Let* $\Omega=\bigcup_i(a_i,b_i)$ *be an open set in* $\mathbb{R}$ *with* $|\Omega|=\sum_i(b_i-a_i)<\infty$. *Let* A *be the number of components of* $\tilde{\Omega}$. *Then we have*

$$\lim_{t\to 0}\frac{|\Omega|-Q_\Omega(t)}{t\ln(1/t)}=\frac{2A}{\pi}.$$

Proof. The proof is very similar to the proof of Theorem 4.2 using Lemmas 4.1, 4.11 and 4.12, 4.13 and we omit the details. □

5 | PERTURBATION RESULTS

By perturbing the Lévy measure, one gets from a familiar Lévy process other interesting Lévy processes. For example, one can use to such a perturbation to get relativistic stable processes from stable processes, see Section 6.4. In this section, we study the stability of the small time asymptotic behavior of the spectral heat content under such perturbations. In this section, we assume that $\mathbf{Y}$ is a Lévy process in $\mathbb{R}^d$ with Lévy triplet (A, γ, ν^Y) such that

$$\lim_{t\to 0} \frac{|\Omega| - Q_\Omega^{\mathbf{Y}}(t)}{f(t)} = C,$$

where $\lim_{t\to 0} \frac{t}{f(t)} = 0$. Throughout this section, the superscript $\mathbf{Y}$ always means quantities corresponding to the process $\mathbf{Y}$.

Now we assume that $\mathbf{X}$ is a Lévy process in $\mathbb{R}^d$ with Lévy triplet (A, γ, ν) such that the signed measure $\sigma(\mathrm{d}x) := \nu(\mathrm{d}x) - \nu^Y(\mathrm{d}x)$ has finite total variation m.

Here is the main result of this section.

Theorem 5.1. *We have*

$$\lim_{t\to 0} \frac{|\Omega| - Q_\Omega(t)}{f(t)} = \lim_{t\to 0} \frac{|\Omega| - Q_\Omega^{\mathbf{Y}}(t)}{f(t)}.$$

In order to prove Theorem 5.1 we need two lemmas.

Lemma 5.2. *If $\sigma(\mathrm{d}x)$ is a nonnegative measure, then for any $t > 0$ we have*

$$|\Omega| - Q_\Omega^{\mathbf{Y}}(t) \le e^{mt}\left(|\Omega| - Q_\Omega(t)\right).$$

Proof. Since $\nu(\mathrm{d}x) = \nu^Y(\mathrm{d}x) + \sigma(\mathrm{d}x)$ and σ is a nonnegative finite measure, we can write $X_t = Y_t + V_t$, where $\mathbf{V} = (V_t)_{t\ge 0}$ is a compound Poisson process independent of $\mathbf{Y}$. Let $T = \inf\{s \ge 0 : V_s \ne 0\}$. It is well known that T is exponentially distributed with parameter $m = \sigma(\mathbb{R}^d)$, see, for instance, [24, Section 19]. Since $\tau_\Omega^{\mathbf{Y}}$ and T are independent, we have

$$\begin{aligned} |\Omega| - Q_\Omega(t) &= \int_\Omega \mathbb{P}_x\left(\tau_\Omega < t\right)\mathrm{d}x \ge \int_\Omega \mathbb{P}_x\left(\tau_\Omega < t, T > t\right)\mathrm{d}x \\ &= \int_\Omega \mathbb{P}_x\left(\tau_\Omega^{\mathbf{Y}} < t, T > t\right)\mathrm{d}x = \int_\Omega \mathbb{P}_x\left(\tau_\Omega^{\mathbf{Y}} < t\right)\mathbb{P}_x(T > t)\,\mathrm{d}x \\ &= \int_\Omega \mathbb{P}_x\left(\tau_\Omega^{\mathbf{Y}} < t\right)e^{-mt}\,\mathrm{d}x = e^{-mt}\left(|\Omega| - Q_\Omega^{\mathbf{Y}}(t)\right). \end{aligned}$$

This establishes the claim of the lemma. □

Lemma 5.3. *If $\sigma(\mathrm{d}x)$ is a nonnegative measure, then for any $t > 0$ we have*

$$|\Omega| - Q_\Omega^{\mathbf{Y}}(t) \ge |\Omega| - e^{mt}Q_\Omega(t) = \left(|\Omega| - Q_\Omega(t)\right) - \left(e^{mt} - 1\right)Q_\Omega(t).$$

Proof. As in the proof of Lemma 5.2, we write $X_t = Y_t + V_t$, where $\mathbf{V}$ is a compound Poisson process independent of $\mathbf{Y}$. Then by independence of $\mathbf{Y}$ and $\mathbf{V}$, we have

$$\begin{aligned} e^{-mt}Q_\Omega^{\mathbf{Y}}(t) &= e^{-mt}\int_\Omega \mathbb{P}_x\left(\tau_\Omega^{\mathbf{Y}} > t\right)\mathrm{d}x = \mathbb{P}_x(T > t)\int_\Omega \mathbb{P}_x\left(\tau_\Omega^{\mathbf{Y}} > t\right)\mathrm{d}x \\ &= \int_\Omega \mathbb{P}_x\left(\tau_\Omega^{\mathbf{Y}} > t, T > t\right)\mathrm{d}x \le \int_\Omega \mathbb{P}_x\left(\tau_\Omega > t\right)\mathrm{d}x = Q_\Omega(t), \end{aligned}$$

where we used the fact that $\{\tau_\Omega^{\mathbf{Y}} > t, T > t\} \subset \{\tau_\Omega > t\}$ in the last inequality. Hence we have $Q_\Omega^{\mathbf{Y}}(t) \le e^{mt}Q_\Omega(t)$ and this immediately implies $|\Omega| - Q_\Omega^{\mathbf{Y}}(t) \ge |\Omega| - e^{mt}Q_\Omega(t)$. □

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. By assumption the signed measure $\sigma(\mathrm{d}x)$ has finite total variation. Let $\sigma(\mathrm{d}x) = \sigma^+(\mathrm{d}x) - \sigma^-(\mathrm{d}x)$ be the Hahn–Jordan decomposition (see [12, Theorem 3.3 and 3.4]) of $\sigma(\mathrm{d}x)$ such that $\mathcal{P} \cup \mathcal{N} = \mathbb{R}^d$, $\mathcal{P} \cap \mathcal{N} = \emptyset$, and

$\sigma^+(\mathcal{N}) = \sigma^-(\mathcal{P}) = 0$. Let $\mathbf{Z}$ be a Lévy process with Lévy density $\nu^{\mathbf{Z}}(dx) = \nu(dx)1_{\mathcal{N}}(x) + \nu^{\mathbf{Y}}(dx)1_{\mathcal{P}}(x)$. Note that $\sigma_+(dx) := \nu(dx) - \nu^{\mathbf{Z}}(dx)$ is a nonnegative measure on $\mathbb{R}^d$ and

$$m_1 := \int_{\mathbb{R}^d} \sigma_+(dx) = \int_{\mathcal{P}} \sigma^+(dx) = \|\sigma^+\| \le \|\sigma\| < \infty.$$

Hence from Lemmas 5.2 and 5.3 we have

$$\begin{aligned} |\Omega| - Q^{\mathbf{Z}}_{\Omega}(t) &\le e^{m_1 t}\left(|\Omega| - Q_{\Omega}(t)\right), \\ |\Omega| - Q^{\mathbf{Z}}_{\Omega}(t) &\ge |\Omega| - Q_{\Omega}(t) - \left(e^{m_1 t} - 1\right)|\Omega|. \end{aligned} \tag{5.1}$$

By interchanging the role of $\mathbf{X}$ and $\mathbf{Y}$ we also have

$$\begin{aligned} |\Omega| - Q^{\mathbf{Z}}_{\Omega}(t) &\le e^{m_2 t}\left(|\Omega| - Q^{\mathbf{Y}}_{\Omega}(t)\right), \\ |\Omega| - Q^{\mathbf{Z}}_{\Omega}(t) &\ge |\Omega| - Q^{\mathbf{Y}}_{\Omega}(t) - \left(e^{m_2 t} - 1\right)|\Omega|, \end{aligned} \tag{5.2}$$

where $\sigma_-(dx) := \nu^{\mathbf{Y}}(dx) - \nu^{\mathbf{Z}}(dx)$ and $m_2 := \int_{\mathbb{R}^d} \sigma_-(dx) = \int_{\mathcal{N}} \sigma^-(dx) < \infty$.

Hence it follows from (5.1) and (5.2) we have

$$|\Omega| - Q^{\mathbf{Y}}_{\Omega}(t) \le |\Omega| - Q^{\mathbf{Z}}_{\Omega}(t) + \left(e^{m_2 t} - 1\right)|\Omega| \le e^{m_1 t}\left(|\Omega| - Q_{\Omega}(t)\right) + \left(e^{m_2 t} - 1\right)|\Omega|,$$

and

$$|\Omega| - Q^{\mathbf{Y}}_{\Omega}(t) \ge e^{-m_2 t}\left(|\Omega| - Q^{\mathbf{Z}}_{\Omega}(t)\right) \ge e^{-m_2 t}\left(|\Omega| - Q_{\Omega}(t) - \left(e^{m_1 t} - 1\right)|\Omega|\right).$$

Since $\lim_{t\to 0} \frac{e^{mt}-1}{f(t)} = 0$, we have

$$\lim_{t\to 0} \frac{|\Omega| - Q^{\mathbf{Y}}_{\Omega}(t)}{f(t)} \le \liminf_{t\to 0} \frac{|\Omega| - Q_{\Omega}(t)}{f(t)},$$

and

$$\lim_{t\to 0} \frac{|\Omega| - Q^{\mathbf{Y}}_{\Omega}(t)}{f(t)} \ge \limsup_{t\to 0} \frac{|\Omega| - Q_{\Omega}(t)}{f(t)}.$$

The proof is now complete. □

6 | EXAMPLES

In this section we examine concrete examples of the small time asymptotic behavior of the spectral heat content for various Lévy processes.

6.1 | Symmetric stable processes in ℝ and their perturbations

Recall that the Lévy measure of the symmetric α-stable process in $\mathbb{R}$ is given by $\nu^{\mathbf{S}^{(\alpha)}}(dx) = \frac{c(1,\alpha)}{|x|^{1+\alpha}}dx$.

Now we assume that $\mathbf{X}$ is a Lévy process in $\mathbb{R}$ with Lévy triplet $(0, 0, \nu)$ such that the signed measure $\sigma(dx) = \nu(dx) - \nu^{\mathbf{S}^{(\alpha)}}(dx)$ has finite total variation m. Let

$$f_\alpha(t) = \begin{cases} t^{1/\alpha} & \text{if } 1 < \alpha \le 2, \\ t \ln \frac{1}{t} & \text{if } \alpha = 1, \\ t & \text{if } 0 < \alpha < 1. \end{cases}$$

Note that, when $0<\alpha<1$, the process $\mathbf{X}$ is of bounded variation. As a consequence of Corollary 3.7, Theorems 4.2, 4.14, and 5.1, we immediately get the following.

Proposition 6.1. *Suppose the assumptions in the paragraph above hold. Let $\Omega=\bigcup_i(a_i,b_i)$ be an open set in $\mathbb{R}$ with $|\Omega|=\sum_i(b_i-a_i)<\infty$. Let A be the number of components of $\tilde{\Omega}$ and B be number of points in $\partial^{ad}\Omega$. Then we have*

$$\lim_{t\to 0}\frac{|\Omega|-Q_\Omega(t)}{f_\alpha(t)}=\begin{cases}2A\mathbb{E}\left[\overline{S^{(\alpha)}}_1\right]+2BC_1, & \text{if } 1<\alpha\le 2,\\ \frac{2A}{\pi}, & \alpha=1,\\ \mathrm{Per}_{\mathbf{X}}(\Omega), & 0<\alpha<1.\end{cases}$$

Remark 6.2. Proposition 6.1 is a natural generalization of the main result in [3]. We remark here that the set $\Omega\subset\mathbb{R}$ is an arbitrary open in $\mathbb{R}$ of finite Lebesgue measure. The class of process we are dealing with here is much larger than the class of symmetric stable processes.

6.2 | Fractional perimeter for symmetric stable processes in $\mathbb{R}$

(i) If $\Omega\subset\mathbb{R}$ has finitely many components, then Ω has a finite perimeter, which is equivalent to $f_\Omega(y)$ is Lipschitz (see [14]), we have $\int_{\mathbb{R}^d}\frac{f_\Omega(y)}{|y|^{d+\alpha}}\,\mathrm{d}y<\infty$. Hence Theorem 3.2 recovers [8, Theorem 3].

(ii) Now we give an example of an open set $\Omega\subset\mathbb{R}$ with $|\Omega|<\infty$ such that $\mathrm{Per}_{\mathbf{S}^{(\alpha)}}(\Omega)=\infty$ for all $0<\alpha<1$. Let $\{d_n\}_{n\in\mathbb{N}}\subset(0,1)$ be a strictly decreasing sequence such that $\sum_{n=1}^\infty d_n<\infty$. We consider an open set $\Omega:=\bigcup_{n=1}^\infty(n,n+d_n)$. Define for each $y\in(0,1)$ a number $n(y)=\sup\{k: d_k\ge y\}$. We have $g_\Omega(y)=\left(d_1+d_2+\cdots+d_{n(y)}\right)-n(y)y$, and

$$f_\Omega(y)=\sum_{k=n(y)+1}^{\infty}d_k+n(y)y\ge n(y)y.$$

Now we fix $b>1$ and consider $d_n=\frac{1}{n(1+\ln n)^b}$. By the definition of $n(y)$ we have

$$\frac{1}{n(y)(1+\ln n(y))^b}\ge y>\frac{1}{(n(y)+1)(1+\ln(n(y)+1))^b}$$

and using this it is easy to see

$$f_\Omega(y)\ge n(y)y\ge\frac{1}{2(2+\ln n(y))^b}\ge c(b)\ln^{-b}\left(y^{-1}\right),\quad y\in(0,1/4). \tag{6.1}$$

Thus for any $0<\alpha<1$ we have

$$\mathrm{Per}_{\mathbf{S}^{(\alpha)}}(\Omega)\ge\int_{\{|y|\le 1/4\}}\frac{f_\Omega(y)}{|y|^{1+\alpha}}\,\mathrm{d}y=\infty.$$

(iii) Finally we state a simple criteria that guarantees $\mathrm{Per}_{\mathbf{S}^{(\alpha)}}(\Omega)<\infty$.

Lemma 6.3. *Let $\alpha\in(0,1)$. Suppose that $\Omega=\bigcup_{i=1}^\infty\Omega_i\subset\mathbb{R}$, where Ω_i are open connected and disjoint. If $\sum_{i=1}^\infty|\Omega_i|^{1-\alpha}<\infty$, then*

$$\mathrm{Per}_{\mathbf{S}^{(\alpha)}}(\Omega)<\infty.$$

Proof. Note that for $x\in\Omega$ we have

$$\int_{\Omega^c}\frac{1}{|x-y|^{1+\alpha}}\,\mathrm{d}y\le 2\int_{\delta_\Omega(x)}^\infty\frac{1}{r^{1+\alpha}}=\frac{2}{\alpha}\delta_\Omega(x)^{-\alpha}.$$

Hence

$$\begin{aligned}\mathrm{Per}_{\mathbf{S}^{(\alpha)}}\Omega &= \int_{\Omega}\int_{\Omega^c}\frac{1}{|x-y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}x \le \int_{\Omega}\frac{2}{\alpha}\delta_\Omega(x)^{-\alpha}\,\mathrm{d}x\\ &= \sum_{i=1}^{\infty}\int_{\Omega_i}\frac{2}{\alpha}\delta_\Omega(x)^{-\alpha}\,\mathrm{d}x = \sum_{i=1}^{\infty}\frac{4}{\alpha}\int_0^{\frac{|\Omega_i|}{2}}\frac{\mathrm{d}r}{r^\alpha} = \sum_{i=1}^{\infty}\frac{4}{\alpha(1-\alpha)}\left(\frac{|\Omega_i|}{2}\right)^{1-\alpha}. \qquad \square\end{aligned}$$

Let consider $\Omega = \bigcup_{n=1}^{\infty}\left(n, n+\frac{1}{n^b}\right)$ with $b > 1$. By the above lemma $\mathrm{Per}_{\mathbf{S}^{(\alpha)}}\Omega < \infty$ if $b > 1/(1-\alpha)$. Hence if $b > 1/(1-\alpha)$, then we have

$$\lim_{t\to 0}\frac{|\Omega| - Q_\Omega(t)}{t} < \infty.$$

On the other hand, using an argument similar to that leading to (6.1), we get $\mathrm{Per}_{\mathbf{S}^{(\alpha)}}\Omega = \infty$, for $b \le 1/(1-\alpha)$. Hence we conclude that

$$\lim_{t\to 0}\frac{|\Omega| - Q_\Omega(t)}{f_\alpha(t)} < \infty \text{ if and only if } b > 1/(1-\alpha).$$

6.3 | Isotropic α-stable processes in $\mathbb{R}^d$, $\alpha \in (0,1)$ and $d \ge 2$

Consider an isotropic α-stable process $\mathbf{S}^{(\alpha)}$ with $0 < \alpha < 1$ on $\mathbb{R}^d$, $d \ge 2$. Suppose that the open set Ω satisfies the following volume density condition (see [33, Equation (1.1)])

$$|\Omega \cap B(x, 2d(x,\partial\Omega))| > c\,\mathrm{dist}(x,\partial D)^d$$

for some constant $c > 0$ and $|\partial\Omega| = 0$. Then it follows from [33, Theorem 1] that $\mathbb{P}_x\left(\tau_\Omega^{\mathbf{S}^{(\alpha)}} \in \partial\Omega\right) = 0$ for all $x \in \Omega$. Hence by Theorem 3.4 we have

$$\lim_{t\to 0}\frac{|\Omega| - Q_\Omega^{\mathbf{S}^{(\alpha)}}(t)}{t} = \mathrm{Per}_{\mathbf{S}^{(\alpha)}}(\Omega).$$

Note that any Lipschitz open sets satisfy volume density condition.

6.4 | Relativistic stable processes

Suppose that $\mathbf{X}^m$ is a relativistic α-stable process with mass $m > 0$ whose characteristic exponent is

$$\psi^m(\xi) = \left(|\xi|^2 + m^{2/\alpha}\right)^{\alpha/2} - m, \quad \xi \in \mathbb{R}^d.$$

Let $\nu^m(x)$ be the Lévy density of $\mathbf{X}^m$. It is well-known that $0 < \nu^m(x) \le \nu^{\mathbf{S}^{(\alpha)}}(x)$ and

$$\int_{\mathbb{R}^d}\left(\nu^{\mathbf{S}^{(\alpha)}}(x) - \nu^m(x)\right)\mathrm{d}x = m < \infty.$$

Hence if $\alpha \ge 1$ and $d = 1$, it follows from Proposition 6.1 that

$$\lim_{t\to 0}\frac{|\Omega| - Q_\Omega^{\mathbf{X}^m}(t)}{f_\alpha(t)} = \begin{cases} 2A\mathbb{E}\left[\overline{S^{(\alpha)}}_1\right] + 2BC_1, & \text{if } 1 < \alpha < 2,\\ \frac{2A}{\pi}, & \alpha = 1.\end{cases}$$

On the other hand when $0 < \alpha < 1$ and Ω is a Lipschitz open set in $\mathbb{R}^d$, $d \ge 2$ or an arbitrary open set in $\mathbb{R}$ it follows from Corollaries 3.5 and 3.7 that

$$\lim_{t\to 0}\frac{|\Omega| - Q_\Omega^{\mathbf{X}^m}(t)}{t} = \mathrm{Per}_{\mathbf{X}^m}(\Omega).$$

6.5 | Truncated stable processes

Let $\mathbf{X}^T$ be a truncated α-stable process with Lévy triplet $\left(0, 0, \nu^{\mathbf{X}^T}\right)$, where

$$\nu^{\mathbf{X}^T}(x) = \nu^{\mathbf{S}^{(\alpha)}}(x) \cdot 1_{\{\|x\| \le 1\}}(x).$$

By the same argument as in the case of relativistic stable processes, we get that when $\alpha \ge 1$ and $d = 1$, we have

$$\lim_{t \to 0} \frac{|\Omega| - Q_\Omega^{\mathbf{X}^T}(t)}{f_\alpha(t)} = \begin{cases} 2A\mathbb{E}\left[\overline{S}_1^{(\alpha)}\right] + 2BC_1, & \text{if } 1 < \alpha < 2, \\ \dfrac{2A}{\pi}, & \alpha = 1. \end{cases}$$

When $0 < \alpha < 1$ and Ω is a Lipschitz open set in $\mathbb{R}^d$, $d \ge 2$ or an arbitrary open set in $\mathbb{R}$ it follows from Corollaries 3.5 and 3.7 that

$$\lim_{t \to 0} \frac{|\Omega| - Q_\Omega^{\mathbf{X}^T}(t)}{t} = \mathrm{Per}_{\mathbf{X}^{\mathrm{T}}}(\Omega).$$

6.6 | Logarithmic perturbations

Let $\mathbf{X}^L$ be a Lévy process with Lévy triplet $\left(0, 0, \nu^{\mathbf{X}^L}\right)$, where

$$\nu^{\mathbf{X}^L}(\mathrm{d}x) = \left(\ln\left(2 + \frac{1}{\|y\|}\right)\right)^\beta \nu^{\mathbf{S}^{(\alpha)}}(\mathrm{d}x),$$

where $\beta \in \mathbb{R}$. By [10, Proposition 2] we have that $\psi^L \in \mathcal{R}_\alpha$ and $\psi^L(s) \sim s^\alpha \ln^\beta(s)$, where $f(s) \sim g(s)$ means $\lim_{s \to \infty} \frac{f(s)}{g(s)} = 1$. This and [6, Proposition 1.5.15] imply $\left(\psi^L\right)^{-1}(s) \sim s^{1/\alpha} \ln^{-\beta/\alpha}(s)$. Hence we get by Theorem 4.2, for $\alpha > 1$

$$\lim_{t \to 0} \frac{|\Omega| - Q_\Omega^{\mathbf{X}^L}(t)}{t^{1/\alpha} \ln^{\beta/\alpha}(1/t)} = 2A\mathbb{E}\left[\overline{S^{(\alpha)}}_1\right] + 2BC_1.$$

When $\alpha < 1$ or $\alpha = 1$ and $\beta < -1$ the process $\mathbf{X}^L$ is of bounded variation, therefore one can apply Theorem 3.4 and Corollaries 3.7 and 3.5 in this case.

6.7 | Strictly stable processes with $0 < \alpha < 1$

Let $\mathbf{X}^{SS}$ be any non-degenerate strictly stable processes with $0 < \alpha < 1$ and Ω be a bounded Lipschitz open set in $\mathbb{R}^d$, $d \ge 1$. The argument in the proof of [24, Theorem 42.30] shows that (3.4) is satisfied. Hence it follows from Corollary 3.6

$$\lim_{t \to 0} \frac{|\Omega| - Q_\Omega^{\mathbf{X}^{SS}}(t)}{t} = \mathrm{Per}_{\mathbf{X}^{SS}}(\Omega).$$

ACKNOWLEDGEMENT

T. Grzywny was supported in part by National Science Centre (Poland): grant 2016/23/B/ST1/01665. R. Song is partially supported by a grant from the Simons Foundation (#429343, Renming Song). We thank the referee for helpful comments on the first version of this paper.

REFERENCES

[1] L. Acuña Valverde, *Heat content estimates over sets of finite perimeter*, J. Math. Anal. Appl. **441** (2016), 104–120.

[2] L. Acuña Valverde, *Heat content for stable processes in domains of* $\mathbb{R}^d$, J. Geom. Anal. **27** (2017), no. 1, 492–524.

[3] L. Acuña Valverde, *On the one dimensional spectral heat content for stable processes*, J. Math. Anal. Appl. **441** (2016), 11–24.

[4] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Math. Monogr., The Clarendon Press, Oxford University Press, New York, 2000.

[5] J. Bertoin, *Lévy processes*, Cambridge University Press, Cambridge, 1996.

[6] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regulary variation*, Encyclopedia Math. Appl., vol. 27, Cambridge University Press, 1987.

[7] J. Bretagnolle, *Résultats de Kesten sur les processus à accroissements indépendants*, Lecture Notes in Math., vol. 191, pp. 21–36.

[8] W. Cygan and T. Grzywny, *Heat content for convolution semigroups*, J. Math. Anal. Appl. **446** (2017), 1393–1414.

[9] W. Cygan and T. Grzywny, *A note on the generalized heat content for Lévy processes*, Bull. Korean Math. Soc. **55** (2018), no. 5, 1463–1481.

[10] W. Cygan, T. Grzywny, and B. Trojan, *Asymptotic behavior of densities of unimodal convolution semigroups*, Trans. Amer. Math. Soc. **369** (2017), no. 8, 5623–5644.

[11] L. C. Evans and R. F Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL, 2015.

[12] G. B. Folland, *Real analysis. Modern techniques and their applications*, 2nd ed., Pure Appl. Math. (N. Y.), A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1999.

[13] N. Fusco, V. Millot, and M. Morini, *A quantitative isoperimetric inequality for fractional perimeters*, J. Funct. Anal. **261** (2011), 697–715.

[14] B. Galerne, *Computation of the perimeter of measurable sets via their covariogram. applications to random sets*, Image Anal. Stereol. **30** (2011), 39–51.

[15] I. I. Gihman and A. V. Skorohod, *The theory of stochastic processes. I*, Springer-Verlag, New York-Heidelberg, 1974.

[16] T. Grzywny, *On Harnack inequality and Hölder regularity for isotropic unimodal Lévy processes*, Potential Anal. **41** (2014), 1–29.

[17] J. Hawkes, *Potential theory of Lévy processes*, Proc. London Math. Soc. **38** (1979), 335–352.

[18] N. Ikeda and S. Watanabe, *On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes*, J. Math. Kyoto Univ. **2** (1962), 79–95.

[19] J. M. Mazon, J. D. Rossi, and J. Toledo, *The heat content for the nonlocal diffusion with non-singular kernels*, Adv. Nonlinear Stud. **17** (2017), no. 2, 255–268.

[20] M. Miranda, Jr., et al., *On a characterisation of perimeters in* $\mathbb{R}^n$ *via heat semigroup*, Ric. Mat. **44** (2005), 615–621.

[21] M. Miranda, Jr., et al., *Short-time heat flow and functions of bounded variation in* $\mathbb{R}^n$, Ann. Fac. Sci. Toulouse Math. (6) **16** (2007), 125–145.

[22] M. Preunkert, *A semigroup version of the isoperimetric inequality*, Semigroup Forum **68** (2004), 233–245.

[23] W. E. Pruitt, *The growth of random walks and Lévy processes*, Ann. Probab. **9** (1981), 948–956.

[24] K. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge Stud. Adv. Math., vol. 68, Cambridge University Press, 1999.

[25] P. Sztonyk, *On harmonic measure for Lévy processes*, Probab. Math. Statist. **20** (2000), 383–390.

[26] S. J. Taylor, *Multiple points for the sample paths of the symmetric stable process*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **5** (1966), 247–264.

[27] M. van den Berg, *Heat content and Brownian motion for some regions with a fractal boundary*, Probab. Theory Related Fields **100** (1994), 439–456.

[28] M. van den Berg, *Heat flow and perimeter in* $\mathbb{R}^m$, Potential Anal. **39** (2013), 369–387.

[29] M. van den Berg, E. B. Dryden, and T. Kappeler, *Isospectrality and heat content*, Bull. Lond. Math. Soc. **46** (2014), 793–808.

[30] M. van den Berg and P. Gilkey, *Heat flow out of a compact manifold*, J. Geom. Anal. **25** (2015), 1576–1601.

[31] M. van den Berg and K. Gittins, *Uniform bounds for the heat content of open sets in Euclidean space*, Differential Geom. Appl. **40** (2015), 67–85.

[32] M. van den Berg and J. F. Le Gall, *Mean curvature and the heat equation*, Math. Z. **215** (1994), 437–464.

[33] J. M. Wu, *Harmonic measures for symmetric stable processes*, Studia Math. **149** (2002), no. 3, 281–293.

[34] K. Yano, Y. Yano, and M. Yor, *On the laws of first hitting times of points for one-dimensional symmetric stable Lévy processes*, Séminaire de Probabilités XLII, vol. 1979, Springer, 2009, pp. 187–227.

How to cite this article: Grzywny T, Park H, Song R. Spectral heat content for Lévy processes. *Mathematische Nachrichten*. 2018;1–21. https://doi.org/10.1002/mana.201800035