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Small time asymptotics of spectral heat contents for subordinate killed Brownian motions related to isotropic α-stable processes

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Abstract

In this paper, we study the small time asymptotic behavior of the spectral heat content $\widetilde{Q}_D^{(\alpha)}(t)$ of an arbitrary bounded $C^{1,1}$ domain D with respect to the *subordinate killed Brownian motion* in D via an $(\alpha/2)$-stable subordinator. For all $\alpha \in (0,2)$, we establish a two-term small time expansion for $\widetilde{Q}_D^{(\alpha)}(t)$ in all dimensions. When $\alpha \in (1,2)$ and $d \geqslant 2$, we establish a three-term small time expansion for $\widetilde{Q}_D^{(\alpha)}(t)$.

1. *Introduction*

Let X be a Markov process in $\mathbb{R}^d$. For any open set $D \subset \mathbb{R}^d$, the heat content of D with respect to X is defined to be

$$H_D^X(t) := \int_D \mathbb{P}_x(X_t \in D)dx,$$

and the spectral heat content of D with respect to X is defined to be

$$Q_D^X(t) := \int_D \mathbb{P}_x(\tau_D^X > t)dx,$$

where τ_D^X is the first time the process X exits D. The spectral heat content of D with respect to X can be regarded as the heat content of D with respect to the killed process X^D. When X is an isotropic α-stable process, $\alpha \in (0,2]$, in $\mathbb{R}^d$, we will write $Q_D^{(\alpha)}(t)$ for $Q_D^X(t)$. In particular, $Q_D^{(2)}(t)$ stands for the spectral heat content of D with respect to Brownian motion.

The heat content with respect to Lévy processes, especially Brownian motions, has been studied extensively, see, for instance, [**1**, **4**, **5**, **12**, **20**, **22**]. The spectral heat content $Q_D^{(2)}(t)$ with respect to Brownian motion has also been studied a lot (see [**2**–**21**]). In [**2**], a two-term small time expansion for $Q_D^{(2)}(t)$ was established for bounded $C^{1,1}$ domains and in [**6**] a three-term small time expansion for $Q_D^{(2)}(t)$ was obtained for bounded domains with C^3 boundary. In [**17**], a recursive formula of the complete asymptotic series of the spectral heat content in a Riemannian manifold with smooth boundary was investigated. The study of the small time asymptotic behavior of the spectral heat content with respect to other Lévy processes is more recent. Upper and lower bounds for $Q_D^{(\alpha)}(t)$, $\alpha \in (0,2)$, were established in [**22**], while explicit expressions for the second term in the asymptotic behavior of $Q_D^{(\alpha)}(t)$, $\alpha \in (0,2)$, in dimension 1 for bounded open intervals were obtained in [**21**]. In the recent paper [**13**], the results of [**21**, **22**] were generalized in several directions.

An isotropic α-stable process $X^{(\alpha)}$ can be obtained from a Brownian motion W via an independent $(\alpha/2)$-stable subordinator $S^{(\alpha/2)}$, that is, $X_t^{(\alpha)} = W_{S_t^{(\alpha/2)}}$. Thus, an isotropic α-stable process is a subordinate Brownian motion. Hence, the spectral heat content $Q_D^{(\alpha)}(t)$

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is the heat content with respect to the *killed subordinate Brownian motion* $X^{(\alpha),D}$, which can be obtained from the Brownian motion W by subordinating with the independent $(\alpha/2)$-stable subordinator $S^{(\alpha/2)}$ first and then killing it upon exiting D. If we reverse the order of the two operations, that is, we first kill the Brownian motion W upon exiting D and then subordinate the killed Brownian motion W^D using the independent $(\alpha/2)$-stable subordinator $S^{(\alpha/2)}$, we get the process $Y_t^{D,(\alpha)} := W^D_{S_t^{(\alpha/2)}}$, which is called a *subordinate killed Brownian motion*. The generator of $X^{(\alpha),D}$ is $-(-\Delta)^{\alpha/2}|_D$, the fractional Laplacian with zero exterior condition, while the generator of $Y^{D,(\alpha)}$ is $-(-\Delta|_D)^{\alpha/2}$, the fractional power of the Dirichlet Laplacian. Subordinate killed Brownian motions are very natural and useful processes. For example, it was used in [**11**] as a tool to obtain two-sided estimates for the eigenvalues of the generator of $X^{(\alpha),D}$. The potential theory of subordinate killed Brownian motions has been studied intensively, see [**15**] and the references therein. In the PDE literature, the operator $-(-\Delta|_D)^{\alpha/2}$ also goes under the name of spectral fractional Laplacian, see [**9**] and the references therein. This operator has been of interest to quite a few people in the PDE circle.

The purpose of this paper is to study the small time asymptotic behavior of the spectral heat content $\widetilde{Q}_D^{(\alpha)}(t)$ with respect to $Y^{D,(\alpha)}$ defined by

$$\widetilde{Q}_D^{(\alpha)}(t) := \int_D \mathbb{P}_x(Y_t^{D,(\alpha)} \in D)dx.$$

The main results of this paper are Theorems 1.1 and 1.2. When dealing with stable processes, the notation $\mathbb{E}$ will stand for the expectation of the process starting from the origin. An open set D in $\mathbb{R}^d$ is said to be a (uniform) $C^{1,1}$ open set if there are (localization radius) $R_0 > 0$ and Λ_0 such that for every $z \in \partial D$ there exist a $C^{1,1}$ function $\psi = \psi_z : \mathbb{R}^d \to \mathbb{R}$ satisfying $\psi(0,\dots,0) = 0$, $\nabla\psi(0) = (0,\dots,0)$, $|\nabla\psi(x) - \nabla\psi(y)| \leqslant \Lambda_0|x-y|$, and an orthonormal coordinate system $CS_z : y = (y_1,\dots,y_{d-1},y_d) := (\tilde{y}, y_d)$ with its origin at z such that $B(z,R_0) \cap D = \{y = (\tilde{y},y_d) \in B(0,R_0) \text{ in } CS_z : y_d > \psi(\tilde{y})\}$. In this paper, we will call the pair (R_0,Λ_0) the characteristics of the $C^{1,1}$ open set D.

THEOREM 1.1. *Let $D = (a,b)$ with $b - a < \infty$ when $d = 1$ or D be a bounded $C^{1,1}$ domain when $d \geqslant 2$. Then*

$$\lim_{t\to 0} \frac{|D| - \widetilde{Q}_D^{(\alpha)}(t)}{f_\alpha(t)} = \begin{cases} \frac{2}{\pi}\Gamma\left(1-\frac{1}{\alpha}\right)|\partial D| = \frac{2|\partial D|}{\sqrt{\pi}}\mathbb{E}\left[\left(S_1^{(\alpha/2)}\right)^{1/2}\right] & \text{if } 1 < \alpha < 2,\\ \frac{2}{\pi}|\partial D| & \text{if } \alpha = 1,\\ \int_0^\infty \left(|D| - Q_D^{(2)}(u)\right)\frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)}u^{-1-\frac{\alpha}{2}}du & \text{if } 0 < \alpha < 1,\end{cases}$$

where $|D|$ is the Lebesgue measure of D, $|\partial D| = 2$ when $d = 1$, $|\partial D|$ is the surface measure of ∂D when $d \geqslant 2$, and

$$f_\alpha(t) = \begin{cases} t^{1/\alpha} & \text{if } \alpha \in (1,2),\\ t\ln\left(\frac{1}{t}\right) & \text{if } \alpha = 1,\\ t & \text{if } \alpha \in (0,1).\end{cases}$$

When $\alpha \in (1,2)$, we identify the third term in the small time expansion of $\widetilde{Q}_D^{(\alpha)}(t)$.

THEOREM 1.2. *Suppose that $d \geqslant 2$, $\alpha \in (1,2)$, and D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. Then the following limit*

$$\lim_{t\to 0} \frac{\widetilde{Q}_D^{(\alpha)}(t) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}}\mathbb{E}\left[\left(S_1^{(\alpha/2)}\right)^{1/2}\right]t^{1/\alpha}\right)}{t}$$

exists and its value is given by

$$\int_0^1 \left(Q_D^{(2)}(u) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}}u^{\frac{1}{2}}\right)\right)\frac{\alpha u^{-1-\frac{\alpha}{2}}}{2\Gamma\left(1-\frac{\alpha}{2}\right)}du$$

$$- \frac{1}{\Gamma\left(1-\frac{\alpha}{2}\right)}\int_D \mathbb{P}_x(\tau_D^{(2)} \leqslant 1)dx + \frac{2|\partial D|\alpha}{\sqrt{\pi}(\alpha-1)\Gamma\left(1-\frac{\alpha}{2}\right)},$$

where $\tau_D^{(2)}$ is the first time the Brownian motion W exits D.

REMARK 1.3. We provide some upper and lower bounds for the constant when D is a ball. It follows from [**17**, Corollary 1.2 (i)] and (3.13) that the constant in Theorem 1.2 is nonnegative when ∂D is smooth. It follows from [**2**, Theorem 6.2] that for a bounded $C^{1,1}$ domain $D \subset \mathbb{R}^d$, $d \geqslant 2$ one has

$$\left|Q_D^{(2)}(u) - \left(|D| - \frac{2|\partial D|u^{1/2}}{\sqrt{\pi}}\right)\right| \leqslant \frac{10^d|D|u}{R_0^2} \quad u > 0, \tag{1.1}$$

where (R_0, Λ_0) is the characteristics of the $C^{1,1}$ domain D. Let $D = B(0,r)$ be a ball with radius r. In this case, we have $R_0 = \frac{r}{2}$. Hence, the constant in Theorem 1.2 is bounded above by

$$\int_0^1 \frac{10^d\frac{\omega_d}{d}r^d u}{(r/2)^2}\frac{\alpha u^{-1-\frac{\alpha}{2}}}{2\Gamma\left(1-\frac{\alpha}{2}\right)}du + \frac{2\alpha\omega_d}{\sqrt{\pi}(\alpha-1)\Gamma\left(1-\frac{\alpha}{2}\right)}$$

$$= \frac{2^2 10^d\alpha\omega_d}{d\Gamma\left(1-\frac{\alpha}{2}\right)(2-\alpha)}r^{d-2} + \frac{2\alpha\omega_d}{\sqrt{\pi}(\alpha-1)\Gamma\left(1-\frac{\alpha}{2}\right)}.$$

On the other hand, by using the trivial bound $\mathbb{P}_x(\tau_D^{(2)} \leqslant 1) \leqslant 1$ and using [**17**, Corollary 1.2 (i)] we see that the constant in Theorem 1.2 is bounded below by

$$-\frac{\omega_d r^d/d}{\Gamma\left(1-\frac{\alpha}{2}\right)} + \frac{2\alpha\omega_d}{\sqrt{\pi}(\alpha-1)\Gamma\left(1-\frac{\alpha}{2}\right)} = \frac{\omega_d}{\Gamma\left(1-\frac{\alpha}{2}\right)}\left(\frac{2\alpha}{\sqrt{\pi}(\alpha-1)} - \frac{r^d}{d}\right).$$

Note that when $r = 1$ we observe that

$$\frac{2\alpha}{\sqrt{\pi}(\alpha-1)} - \frac{1}{d} > \frac{4}{\sqrt{\pi}} - \frac{1}{d} > 0 \quad \text{for all } \alpha \in (1,2).$$

Note that, similar to the case of Brownian motion, the first term in the small time expansion of $\widetilde{Q}_D^{(\alpha)}(t)$ in Theorem 1.1 involves the volume of the domain D and the second term is related to the perimeter $|\partial D|$ of D. In Proposition 3.5, we will show that the second terms in the small time expansions of $\widetilde{Q}_D^{(\alpha)}(t)$ and of $Q_D^{(\alpha)}(t)$ are different when $\alpha \in (1,2)$ and D is a bounded open interval in $\mathbb{R}^1$.

In the Brownian motion case, the third term in the expansion of $Q_D^{(2)}(t)$ involves the mean curvature of D. However, the third term in the small time expansion of $\widetilde{Q}_D^{(\alpha)}(t)$ in Theorem 1.2 is given by a nonexplicit expression. This is probably unavoidable. See the heuristic explanation after the proof of Theorem 1.2.

The organization of the paper is as follows. In Section 2, we fix our notation and recall some basic facts for later use. In Section 3, the main results, Theorems 1.1 and 1.2, are proved.

In this paper, we use the convention that c, lower case or capital, and with or without subscript, stands for a constant whose value is not important and may change from one appearance to another.

2. *Preliminaries*

We first collect some basic facts about stable subordinators. Recall that, for any $\alpha \in (0,2)$, an $(\alpha/2)$-stable subordinator $S_t^{(\alpha/2)}$ is a nondecreasing Lévy process with $S_0^{(\alpha/2)} = 0$ and

$$\mathbb{E}\left[e^{-\lambda S_t^{(\alpha/2)}}\right] = e^{-t\lambda^{\alpha/2}}, \quad \lambda > 0, t \geqslant 0. \tag{2.1}$$

It is well known that the characteristic exponent of an $(\alpha/2)$-stable subordinator is given by

$$\Psi(\theta) = |\theta|^{\frac{\alpha}{2}}\Big(\cos\frac{\pi\alpha}{4} - i\sin\frac{\pi\alpha}{4}\mathrm{sgn}\theta\Big). \tag{2.2}$$

It follows from (2.1) that $S_t^{(\alpha/2)}$ and $t^{2/\alpha}S_1^{(\alpha/2)}$ have the same distribution. The subordinator $S^{(\alpha/2)}$ has a continuous transition density $g^{(\alpha/2)}(t,x)$. It follows from [**10**, (18)] that $g^{(\alpha/2)}(1,x)$ is given by

$$g^{(\alpha/2)}(1,x) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\Gamma(1+\frac{\alpha n}{2})}{n!}\sin\Big(\frac{\pi\alpha n}{2}\Big)x^{-\frac{\alpha n}{2}-1}, \quad x > 0. \tag{2.3}$$

It follows from Stirling's formula

$$\Gamma(1+z) \sim \sqrt{2\pi z}\Big(\frac{z}{e}\Big)^z, \quad z \to \infty,$$

that for any $\varepsilon > 0$ there exists N such that for all $n \geqslant N$ we have

$$\frac{\Gamma(1+\frac{\alpha n}{2})}{n!} \leqslant \frac{1+\varepsilon}{1-\varepsilon}\frac{\sqrt{2\pi\frac{\alpha n}{2}}\Big(\frac{\frac{\alpha n}{2}}{e}\Big)^{\frac{\alpha n}{2}}}{\sqrt{2\pi n}\big(\frac{n}{e}\big)^n} = \frac{1+\varepsilon}{1-\varepsilon}\Big(\frac{\alpha}{2}\Big)^{\frac{\alpha}{2}(n+1)}\Big(\frac{n}{e}\Big)^{-\left(1-\frac{\alpha}{2}\right)n}. \tag{2.4}$$

Thus,

$$\sum_{n=N}^{\infty}\left|(-1)^{n+1}\frac{\Gamma(1+\frac{\alpha n}{2})}{n!}\sin\Big(\frac{\pi\alpha n}{2}\Big)x^{-\frac{\alpha n}{2}-1}\right| \leqslant \sum_{n=N}^{\infty}\frac{1+\varepsilon}{1-\varepsilon}\Big(\frac{\alpha}{2}\Big)^{\frac{\alpha}{2}(n+1)}\Big(\frac{n}{e}\Big)^{-\left(1-\frac{\alpha}{2}\right)n} < \infty.$$

Hence the infinite series in (2.3) converges absolutely for all $x > 0$. Using (2.3), Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \notin \mathbb{Z},$$

and the absolute convergence of the series in (2.3), we get

$$\lim_{x\to\infty} g^{(\alpha/2)}(1,x)x^{1+\frac{\alpha}{2}} = \frac{\Gamma(1+\frac{\alpha}{2})\sin\big(\frac{\pi\alpha}{2}\big)}{\pi} = \frac{\alpha}{2\Gamma\big(1-\frac{\alpha}{2}\big)}. \tag{2.5}$$

(We note in passing that the constant given in [**8**, (5.18)] is incorrect, due to some typos in transcribing the formula from [**18**].) By the scaling property, the transition density $g^{(\alpha/2)}(t,x)$ is equal to $t^{-2/\alpha}g^{(\alpha/2)}(1,\frac{x}{t^{2/\alpha}})$. It follows from (2.2) and the inverse Fourier transform that for all $x > 0$,

$$
\begin{aligned}
g^{(\alpha/2)}(1,x) &= (2\pi)^{-1/2}\int_{\mathbb{R}} e^{i\theta x}e^{-|\theta|^{\alpha/2}\left(\cos\frac{\pi\alpha}{4}-i\sin\frac{\pi\alpha}{4}\mathrm{sgn}\theta\right)}d\theta \\
&\leqslant (2\pi)^{-1/2}\int_{\mathbb{R}} \left|e^{i\theta x}e^{-|\theta|^{\alpha/2}\left(\cos\frac{\pi\alpha}{4}-i\sin\frac{\pi\alpha}{4}\mathrm{sgn}\theta\right)}\right|d\theta \\
&\leqslant (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-|\theta|^{\alpha/2}\cos\frac{\pi\alpha}{4}}d\theta < \infty. \qquad (2.6)
\end{aligned}
$$

On the other hand, when $x \geqslant 1$ it follows from (2.3) and (2.4) that

$$
\begin{aligned}
g^{(\alpha/2)}(1,x) &= x^{-\frac{\alpha}{2}-1}\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\Gamma(1+\frac{\alpha n}{2})}{n!}\sin\left(\frac{\alpha\pi n}{2}\right)x^{-\frac{\alpha(n-1)}{2}} \\
&\leqslant x^{-1-\frac{\alpha}{2}}\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\Gamma(1+\frac{\alpha n}{2})}{n!}x^{-\frac{\alpha(n-1)}{2}} \\
&\leqslant x^{-1-\frac{\alpha}{2}}\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\Gamma(1+\frac{\alpha n}{2})}{n!} \leqslant c_1 x^{-1-\frac{\alpha}{2}}. \qquad (2.7)
\end{aligned}
$$

Hence, it follows from (2.6) and (2.7) that there exists a constant $c_2 > 0$ such that

$$
g^{(\alpha/2)}(t,x) \leqslant c_2\left(t^{-2/\alpha}\wedge\frac{t}{x^{1+\frac{\alpha}{2}}}\right), \quad x>0. \qquad (2.8)
$$

We remark here that in case of symmetric stable processes, the transition density also has a matching lower bound of the form (2.8). However, in the case of stable subordinators, a lower bound similar to the right-hand side of (2.8) does not hold (see [**14**, Lemma 1]).

The following fact is from [**22**, Proposition 2.1] and will be used in the next section.

$$
\mathbb{E}\left[\left(S_1^{(\alpha/2)}\right)^{\gamma}\right] = \frac{\Gamma(1-\frac{2\gamma}{\alpha})}{\Gamma(1-\gamma)}, \quad -\infty<\gamma<\frac{\alpha}{2}. \qquad (2.9)
$$

Now we proceed to define for D the killed subordinate Brownian motion and subordinate killed Brownian motion with respect to an isotropic α-stable process and their respective spectral heat contents. Let W be a Brownian motion in $\mathbb{R}^d$ with generator Δ and $S_t^{(\alpha/2)}$ an $(\alpha/2)$-stable subordinator independent of W. Then, the subordinate Brownian motion $X^{(\alpha)}$ defined by

$$
X_t^{(\alpha)} := W_{S_t^{(\alpha/2)}}, \quad t \geqslant 0,
$$

is an isotropic α-stable process. For any domain $D\subset\mathbb{R}^d$, the process $X^{(\alpha),D}$ defined by

$$
X_t^{(\alpha),D} := \begin{cases} X_t^{(\alpha)} & \text{if } t<\tau_D^{(\alpha)}, \\ \partial & \text{if } t\geqslant\tau_D^{(\alpha)}, \end{cases} \quad t\geqslant 0,
$$

where

$$
\tau_D^{(\alpha)} := \inf\{t>0 : X_t^{(\alpha)}\notin D\}, \quad \alpha\in(0,2],
$$

and ∂ is a point not contained in D (the cemetery point), is called a killed subordinate Brownian motion, or more precisely, a killed isotropic α-stable process in D. When $\alpha=2$, $X_t^{(2)}$ will be a Brownian motion that will be simply denoted by W. The spectral heat content of D with respect to $X^{(\alpha)}$ is defined to be

$$
Q_D^{(\alpha)}(t) := \int_D \mathbb{P}_x\left(\tau_D^{(\alpha)}>t\right)dx.
$$

Now let W^D be the killed Brownian motion in D. The subordinate killed Brownian motion $Y^{D,(\alpha)}$ is defined by

$$Y_t^{D,(\alpha)} := W^D_{S_t^{(\alpha/2)}}, \quad t \geqslant 0.$$

Let ζ^α be the lifetime of $Y^{D,(\alpha)}$, which is the same as the first time the process $Y^{D,(\alpha)}$ exits D. The spectral heat content of D with respect to $Y^{D,(\alpha)}$ is defined by

$$\widetilde{Q}_D^{(\alpha)}(t) := \int_D \mathbb{P}_x\left(Y_t^{D,(\alpha)} \in D\right) dx = \int_D \mathbb{P}_x(\zeta^\alpha > t) dx, \quad t > 0.$$

Note that

$$\{\zeta^\alpha > t\} = \left\{\tau_D^{(2)} > S_t^{(\alpha/2)}\right\}, \quad t > 0.$$

Hence,

$$\widetilde{Q}_D^{(\alpha)}(t) = \int_D \mathbb{P}_x\left(\tau_D^{(2)} > S_t^{(\alpha/2)}\right) dx, \quad t > 0.$$

Note that the following simple relationship is valid

$$\{\zeta^\alpha > t\} = \left\{\tau_D^{(2)} > S_t^{(\alpha/2)}\right\} \subset \left\{\tau_D^{(\alpha)} > t\right\}, \quad t > 0,$$

which in turn implies

$$\widetilde{Q}_D^{(\alpha)}(t) = \int_D \mathbb{P}_x(\zeta^\alpha > t) dx \leqslant \int_D \mathbb{P}_x\left(\tau_D^{(\alpha)} > t\right) dx = Q_D^{(\alpha)}(t), \quad t > 0. \tag{2.10}$$

We end this section by paraphrasing the explanation given on [**19**, p. 579] about the difference between the processes $X_t^{(\alpha),D}$ and $Y_t^{(\alpha),D}$. Look at a path of the Brownian motion W in $\mathbb{R}^d$, and put a mark on it at all the times given by the subordinator $S_t^{(\alpha/2)}$. In this way we observe a trajectory of the process $X_t^{(\alpha)}$. The corresponding trajectory of $Y_t^{(\alpha),D}$ is given by all the marks on the Brownian path prior to $\tau_D^{(2)}$. There is a first mark on the Brownian path following the exit time $\tau_D^{(2)}$. If this mark happens to be in D, the process $X_t^{(\alpha)}$ has not been killed yet, and the mark corresponds to a point on the trajectory of $X_t^{(\alpha),D}$, but not to a point on the trajectory of $Y_t^{(\alpha),D}$. If, on the other hand, the first mark on the Brownian path following the exit time $\tau_D^{(2)}$ happens to be in D^c, then trajectories of $Y_t^{(\alpha),D}$ and $X_t^{(\alpha),D}$ are equal. See the picture on [**19**, p. 581] for an illustration.

3. *Proofs of the main results*

In this section, we prove the main results of this paper, Theorem 1.1 and Theorem 1.2. First we deal with the case for $\alpha \in (1,2)$.

PROPOSITION 3.1. *Let $\alpha \in (1,2)$. Suppose that D is a bounded open interval when $d = 1$ or a bounded $C^{1,1}$ domain when $d \geqslant 2$. Then,*

$$\lim_{t\to 0} \frac{|D| - \widetilde{Q}_D^{(\alpha)}(t)}{t^{1/\alpha}} = \frac{2|\partial D|}{\sqrt{\pi}} \mathbb{E}\left[\left(S_1^{(\alpha/2)}\right)^{1/2}\right].$$

Proof. Note that it follows from [**2**, Theorem 6.2; **21**, Theorem 1.1] that

$$\lim_{t\downarrow 0} \frac{|D| - Q_D^{(2)}(t)}{t^{1/2}} = \frac{2}{\sqrt{\pi}}|\partial D|. \tag{3.1}$$

It follows from (3.1) there exists $\eta > 0$ such that

$$\frac{|D| - Q_D^{(2)}(t)}{\sqrt{t}} \leqslant 1 + \frac{2|\partial D|}{\sqrt{\pi}}, \quad t \in (0, \eta]$$

and if $t \geqslant \eta$, we have that

$$\frac{|D| - Q_D^{(2)}(t)}{\sqrt{t}} \leqslant \frac{|D|}{\sqrt{\eta}}.$$

Taking $C := \max\left\{1 + \frac{2|\partial D|}{\sqrt{\pi}}, \frac{|D|}{\sqrt{\eta}}\right\}$, we get

$$\frac{|D| - Q_D^{(2)}(t)}{\sqrt{t}} \leqslant C, \quad \text{for all } t > 0. \tag{3.2}$$

By the scaling property of $S_t^{(\alpha/2)}$ and Fubini's theorem, we have

$$\begin{aligned}
|D| - \widetilde{Q}_D^{(\alpha)}(t) &= \int_D \mathbb{P}_x\left(\tau_D^{(2)} \leqslant S_t^{(\alpha/2)}\right) dx = \int_D \mathbb{P}_x\left(\tau_D^{(2)} \leqslant t^{2/\alpha} S_1^{(\alpha/2)}\right) dx \\
&= \int_D \int_0^\infty \mathbb{P}_x\left(\tau_D^{(2)} \leqslant t^{2/\alpha} u\right) g^{(\alpha/2)}(1,u) du dx = \int_0^\infty \int_D \mathbb{P}_x\left(\tau_D^{(2)} \leqslant t^{2/\alpha} u\right) dx g^{(\alpha/2)}(1,u) du \\
&= \int_0^\infty \left(|D| - Q_D^{(2)}(t^{2/\alpha} u)\right) g^{(\alpha/2)}(1,u) du = \int_0^\infty \left(\frac{|D| - Q_D^{(2)}(t^{2/\alpha} u)}{t^{1/\alpha} u^{1/2}}\right) t^{1/\alpha} u^{1/2} g^{(\alpha/2)}(1,u) du.
\end{aligned}$$

Hence, it follows from (3.1), (3.2), (2.9) and the Lebesgue dominated convergence theorem that

$$\lim_{t\to 0} \frac{|D| - \widetilde{Q}_D^{(\alpha)}(t)}{t^{1/\alpha}} = \int_0^\infty \lim_{t\to 0}\left(\frac{|D| - Q_D^{(2)}(t^{2/\alpha} u)}{t^{1/\alpha} u^{1/2}}\right) u^{1/2} g^{(\alpha/2)}(1,u) du = \frac{2|\partial D|}{\sqrt{\pi}} \mathbb{E}\left[\left(S_1^{(\alpha/2)}\right)^{1/2}\right].$$

□

Next we deal with the case for $\alpha = 1$. We need the following simple lemma.

Lemma 3.2. *For any $\delta > 0$, we have*

$$\lim_{t\downarrow 0} \frac{\mathbb{E}\left[\left(S_1^{(1/2)}\right)^{\frac{1}{2}}, 0 < S_1^{(1/2)} < \frac{\delta}{t^2}\right]}{\ln(1/t)} = \frac{1}{\sqrt{\pi}}.$$

Proof. It follows from (2.9) and an application of Fatou's lemma that

$$\lim_{t\downarrow 0} \mathbb{E}\left[\left(S_1^{(1/2)}\right)^{\frac{1}{2}}, 0 < S_1^{(1/2)} < \frac{\delta}{t^2}\right] = \infty.$$

Hence, it follows from L'Hôpital's rule, (2.5), and the change of variables $x = \delta t^{-2}$ that

$$\begin{aligned}
\lim_{t\downarrow 0} \frac{\mathbb{E}\left[\left(S_1^{(1/2)}\right)^{\frac{1}{2}}, 0 < S_1^{(1/2)} < \frac{\delta}{t^2}\right]}{\ln(1/t)} &= \lim_{t\downarrow 0} \frac{\int_0^{\delta/t^2} u^{1/2} g^{(1/2)}(1,u) du}{\ln(1/t)} \\
&= \lim_{t\downarrow 0} \frac{(\delta t^{-2})^{1/2} g^{(1/2)}(1, \delta t^{-2})(-2)\delta t^{-3}}{-1/t} = \lim_{t\downarrow 0} 2 g^{(1/2)}(1, \delta t^{-2})(\delta t^{-2})^{3/2} \\
&= \lim_{x\uparrow\infty} 2 g^{(1/2)}(1,x) x^{3/2} = \frac{1}{\sqrt{\pi}}.
\end{aligned}$$

□

Proposition 3.3. *Let $\alpha = 1$. Suppose that D is a bounded open interval when $d = 1$ or a bounded $C^{1,1}$ domain when $d \geqslant 2$. Then,*

$$\lim_{t\to 0} \frac{|D| - \widetilde{Q}_D^{(1)}(t)}{t\ln(\frac{1}{t})} = \frac{2|\partial D|}{\pi}.$$

Proof. As in the proof of Proposition 3.1. we have

$$|D| - \widetilde{Q}_D^{(1)}(t) = \int_0^\infty \left(\frac{|D| - Q_D^{(2)}(t^2u)}{tu^{1/2}}\right) tu^{1/2} g^{(1/2)}(1,u)du$$

$$= \int_0^{\delta t^{-2}} \left(\frac{|D| - Q_D^{(2)}(t^2u)}{tu^{1/2}}\right) tu^{1/2} g^{(1/2)}(1,u)du + \int_{\delta t^{-2}}^\infty \left(|D| - Q_D^{(2)}(t^2u)\right) g^{(1/2)}(1,u)du, \tag{3.3}$$

where the value of δ will be determined later.

For any $\varepsilon > 0$, it follows from (3.1) that there exists $\delta > 0$ such that

$$\frac{2|\partial D|}{\sqrt{\pi}} - \varepsilon < \frac{|D| - Q_D^{(2)}(t)}{\sqrt{t}} < \frac{2|\partial D|}{\sqrt{\pi}} + \varepsilon, \quad t < \delta.$$

For this choice of δ it follows from Lemma 3.2 that

$$\limsup_{t\to 0} \frac{\int_0^{\delta t^{-2}} \left(\frac{|D| - Q_D^{(2)}(t^2u)}{tu^{1/2}}\right) tu^{1/2} g^{(1/2)}(1,u)du}{t\ln(1/t)} \leqslant \left(\frac{2|\partial D|}{\sqrt{\pi}} + \varepsilon\right)\frac{1}{\sqrt{\pi}}. \tag{3.4}$$

Similarly we have

$$\liminf_{t\to 0} \frac{\int_0^{\delta t^{-2}} \left(\frac{|D| - Q_D^{(2)}(t^2u)}{tu^{1/2}}\right) tu^{1/2} g^{(1/2)}(1,u)du}{t\ln(1/t)} \geqslant \left(\frac{2|\partial D|}{\sqrt{\pi}} - \varepsilon\right)\frac{1}{\sqrt{\pi}}. \tag{3.5}$$

For the second term in (3.3), we get from (2.8) and the fact $|D| - Q_D^{(2)}(t) \leqslant |D|$ for all $t > 0$ that

$$\int_{\delta t^{-2}}^\infty \left(|D| - Q_D^{(2)}(t^2u)\right) g^{(1/2)}(1,u)du \leqslant c_1|D| \int_{\delta t^{-2}}^\infty u^{-3/2}du = c_2|D|t$$

for some constants c_1 and c_2. This implies that

$$\limsup_{t\to 0} \frac{\int_{\delta t^{-2}}^\infty \left(|D| - Q_D^{(2)}(t^2u)\right) g^{(1/2)}(1,u)du}{t\ln(1/t)} = 0. \tag{3.6}$$

Since ε is arbitrary, the conclusion of the proposition follows from (3.4), (3.5), and (3.6). □

Finally, we deal with the case for $\alpha \in (0,1)$.

Proposition 3.4. *Let $\alpha \in (0,1)$. Suppose that D is a bounded open interval when $d = 1$ or a bounded $C^{1,1}$ domain when $d \geqslant 2$. Then, we have*

$$\lim_{t\downarrow 0} \frac{|D| - \widetilde{Q}_D^{(\alpha)}(t)}{t} = \int_0^\infty \left(|D| - Q_D^{(2)}(u)\right) \frac{\alpha}{2\Gamma(1-\frac{\alpha}{2})} u^{-1-\frac{\alpha}{2}} du.$$

Proof. Note that by Fubini's theorem,

$$\frac{1}{t}\left(|D|-\widetilde{Q}^{(\alpha)}_D(t)\right)=\frac{1}{t}\int_D \mathbb{P}_x\left(\tau^{(\alpha)}_D\leqslant S^{(\alpha/2)}_t\right)dx=\int_0^\infty\left(|D|-Q^{(\alpha)}_D(u)\right)\frac{g^{(\alpha/2)}(t,u)}{t}du.$$

When $u\geqslant 1$, it follows from (2.8) that

$$\left(|D|-Q^{(2)}_D(u)\right)\frac{g^{(\alpha/2)}(t,u)}{t}\leqslant c_1|D|u^{-1-\frac{\alpha}{2}}. \tag{3.7}$$

On the other hand, when $0<u<1$, it follows from (3.2) that $|D|-Q^{(2)}_D(u)\leqslant Cu^{\frac{1}{2}}$. Hence, from (2.8) we have

$$\left(|D|-Q^{(2)}_D(u)\right)\frac{g^{(\alpha/2)}(t,u)}{t}\leqslant Cu^{-\frac{1}{2}-\frac{\alpha}{2}}. \tag{3.8}$$

Let $\varepsilon>0$ and $\phi_\varepsilon\in C_b(\mathbb{R}^1)$ be such that $1_{B(0,\varepsilon)^c}\leqslant\phi_\varepsilon\leqslant 1_{B(0,\frac{\varepsilon}{2})^c}$ so that the function $u\to(|D|-Q^{(2)}_D(u))\phi_\varepsilon(u)g^{(\alpha/2)}(t,u)/t$ is bounded, continuous, and vanishes near zero. Since $\alpha\in(0,1)$ it follows from [**16**, Corollary 8.9] and the Lebesgue dominated convergence theorem for any $\eta>0$ there exists $t_0>0$ such that

$$\left|\int_0^\infty\left(|D|-Q^{(2)}_D(u)\right)\phi_\varepsilon(u)\frac{g^{(\alpha/2)}(t,u)}{t}du-\int_0^\infty\left(|D|-Q^{(2)}_D(u)\right)\phi_\varepsilon(u)\frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)}u^{-1-\frac{\alpha}{2}}du\right|<\eta \tag{3.9}$$

for all $t\leqslant t_0$. It follows from (3.7), (3.8), and the Lebesgue dominated convergence theorem we have

$$\lim_{\varepsilon\to 0}\int_0^\infty\left(|D|-Q^{(2)}_D(u)\right)\phi_\varepsilon(u)\frac{g^{(\alpha/2)}(t,u)}{t}du=\int_0^\infty\left(|D|-Q^{(2)}_D(u)\right)\frac{g^{(\alpha/2)}(t,u)}{t}du$$

uniformly for all $t\leqslant t_0$. Finally letting $\varepsilon\to 0$ and using the Lebesgue dominated convergence theorem in (3.9), we arrive at the conclusion of the proposition. □

Proof of Theorem 1.1. The proof is an easy consequence of Propositions 3.1, 3.3, and 3.4. □

The second term of the asymptotic expansion of $Q^{(\alpha)}_D(t)$ is known when $d=1$ and D is a bouned open interval, see [**21**]. We now show that the second terms in the expansions of $\widetilde{Q}^{(\alpha)}_D(t)$ and of $Q^{(\alpha)}_D(t)$ are different when $d=1$ and $\alpha\in(1,2)$.

PROPOSITION 3.5. *Suppose that $1<\alpha<2$ and $D\subset\mathbb{R}^1$ is a bounded open interval. Then,*

$$\lim_{t\to 0}\frac{|D|-Q^{(\alpha)}_D(t)}{t^{1/\alpha}}<\lim_{t\to 0}\frac{|D|-\widetilde{Q}^{(\alpha)}_D(t)}{t^{1/\alpha}}.$$

Proof. It is proved in [**21**, Theorem 1.1] that for $\alpha\in(1,2)$,

$$\lim_{t\to 0}\frac{|D|-Q^{(\alpha)}_D(t)}{t^{1/\alpha}}=2\mathbb{E}\left[\overline{X}^{(\alpha)}_1\right].$$

It follows from [**21**, Proposition 2.1] that

$$\mathbb{P}\left(u\leqslant X^{(\alpha)}_t\right)\leqslant\mathbb{P}\left(u\leqslant\overline{X}^{(\alpha)}_t\right)\leqslant 2\mathbb{P}\left(u\leqslant X^{(\alpha)}_t\right).$$

This implies that

$$\mathbb{E}\left[X_1^{(\alpha)}, X_1^{(\alpha)} > 0\right] = \int_0^\infty \mathbb{P}\left(X_1^{(\alpha)} \geqslant u\right) du \leqslant \mathbb{E}\left[\overline{X_1}^{(\alpha)}\right] = \int_0^\infty \mathbb{P}\left(\overline{X_1}^{(\alpha)} \geqslant u\right) du$$

$$\leqslant \int_0^\infty 2\mathbb{P}\left(X_1^{(\alpha)} \geqslant u\right) du = 2\mathbb{E}\left[X_1^{(\alpha)}, X_1^{(\alpha)} > 0\right]. \tag{3.10}$$

It is shown in [22, p. 11] that $\mathbb{E}[X_1^{(\alpha)}, X_1^{(\alpha)} > 0] = \frac{1}{\pi}\Gamma(1-\frac{1}{\alpha})$ and this implies together with Theorem 1.1 that

$$\lim_{t\to 0} \frac{|D| - Q_D^{(\alpha)}(t)}{t^{1/\alpha}} = 2\mathbb{E}[\overline{X}_1^{(\alpha)}] \leqslant \frac{4}{\pi}\Gamma\left(1-\frac{1}{\alpha}\right) = \lim_{t\to 0} \frac{|D| - \widetilde{Q}_D^{(\alpha)}(t)}{t^{1/\alpha}}.$$

Now we assume that $\lim_{t\to 0} \frac{|D|-Q_D^{(\alpha)}(t)}{t^{1/\alpha}} = \lim_{t\to 0} \frac{|D|-\widetilde{Q}_D^{(\alpha)}(t)}{t^{1/\alpha}}$. Then, this would imply by (3.10) that

$$2\int_0^\infty \mathbb{P}(\overline{X}_1^{(\alpha)} \geqslant u)du = 2\mathbb{E}\left[\overline{X_1}^{(\alpha)}\right]$$

$$= 4\mathbb{E}\left[X_1^{(\alpha)}, X_1^{(\alpha)} \geqslant 0\right] = 4\int_0^\infty \mathbb{P}(X_1^{(\alpha)} \geqslant u)du,$$

which would imply that $\mathbb{P}(u \leqslant \overline{X_1}^{(\alpha)}) = 2\mathbb{P}(u \leqslant X_1^{(\alpha)})$ for almost every $u > 0$. But this contradicts [7, Proposition VIII.4] which says that

$$\lim_{u\to\infty} \frac{\mathbb{P}\left(\overline{X}_1^{(\alpha)} \geqslant u\right)}{\mathbb{P}\left(X_1^{(\alpha)} \geqslant u\right)} = 1.$$

□

Combining Theorem 1.1 with (2.10), we immediately get the following: when $d \geqslant 2$, D is a bounded $C^{1,1}$ domain and $\alpha \in (1,2)$, we have

$$\limsup_{t\to 0} \frac{|D| - Q_D^{(\alpha)}(t)}{t^{1/\alpha}} \leqslant \frac{2}{\pi}\Gamma\left(1-\frac{1}{\alpha}\right)|\partial D|, \tag{3.11}$$

and when $\alpha = 1$

$$\limsup_{t\to 0} \frac{|D| - Q_D^{(1)}(t)}{t\ln(\frac{1}{t})} \leqslant \frac{2}{\pi}|\partial D|. \tag{3.12}$$

Comparing (3.11) and (3.12) with [22, Theorem 1.3], we observe that (3.11) and (3.12) are better upper bounds. We remark that it is conjectured in [22] that the limits in (3.7) and (3.8) actually exist but this problem is still open.

Now we establish a three-term small time asymptotic expansion for $\widetilde{Q}_D^{(\alpha)}(t)$ when $\alpha \in (1,2)$. First we need the following simple fact.

Lemma 3.6. *Let $\alpha \in (1,2)$. Then,*

$$\lim_{t\to 0} \frac{\mathbb{E}\left[\left(S_1^{(\alpha/2)}\right)^{k/2}, 0 < S_1^{(\alpha/2)} < t^{-2/\alpha}\right]}{t^{1-\frac{k}{\alpha}}} = \frac{\alpha}{(k-\alpha)\Gamma(1-\frac{\alpha}{2})}, \quad k \geqslant 2.$$

Proof. It follows from (2.9) that both the numerator and the denominator diverge to ∞ as $t \to 0$ when $k \geqslant 2$. Hence, it follows from L'Hôpital's rule and (2.5) that

$$\lim_{t\to 0} \frac{\mathbb{E}\left[\left(S_1^{(\alpha/2)}\right)^{k/2}, 0 < S_1^{(\alpha/2)} < t^{-2/\alpha}\right]}{t^{1-\frac{k}{\alpha}}} = \lim_{t\to 0} \frac{\int_0^{t^{-2/\alpha}} s^{k/2} g^{(\alpha/2)}(1,s)ds}{t^{1-\frac{k}{\alpha}}}$$

$$= \lim_{t\to 0} \frac{t^{-k/\alpha} g^{(\alpha/2)}(1, t^{-2/\alpha}) \frac{-2}{\alpha} t^{-\frac{2}{\alpha}-1}}{(1-\frac{k}{\alpha}) t^{-\frac{k}{\alpha}}} = \lim_{t\to 0} \frac{2}{k-\alpha} g^{(\alpha/2)}(1, t^{-2/\alpha}) t^{-1-\frac{2}{\alpha}} = \frac{\alpha}{(k-\alpha)\Gamma(1-\frac{\alpha}{2})}.$$

□

Note that we have

$$\begin{aligned}
&\widetilde{Q}_D^{(\alpha)}(t) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} \mathbb{E}\left[\left(S_1^{\alpha/2}\right)^{1/2}\right] t^{1/\alpha}\right) \\
&= \int_D \mathbb{P}_x\left(\tau_D^{(2)} \leqslant S_t^{(\alpha/2)}\right) dx - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} \mathbb{E}\left[\left(S_1^{\alpha/2}\right)^{1/2}\right] t^{1/\alpha}\right) \\
&= \int_D \int_0^\infty \mathbb{P}_x\left(\tau_D^{(2)} \leqslant s t^{2/\alpha}\right) g^{(\alpha/2)}(1,s) ds dx - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} \mathbb{E}\left[\left(S_1^{\alpha/2}\right)^{1/2}\right] t^{1/\alpha}\right) \\
&= \int_0^\infty \left(\int_D \mathbb{P}_x\left(\tau_D^{(2)} \leqslant s t^{2/\alpha}\right) dx - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} s^{1/2} t^{1/\alpha}\right)\right) g^{(\alpha/2)}(1,s) ds \\
&= \int_0^\infty \left(Q_D^{(2)}(s t^{2/\alpha}) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} s^{1/2} t^{1/\alpha}\right)\right) g^{(\alpha/2)}(1,s) ds \\
&= \int_0^{t^{-2/\alpha}} \left(Q_D^{(2)}(s t^{2/\alpha}) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} s^{1/2} t^{1/\alpha}\right)\right) g^{(\alpha/2)}(1,s) ds \\
&\quad + \int_{t^{-2/\alpha}}^\infty \left(Q_D^{(2)}(s t^{2/\alpha}) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} s^{1/2} t^{1/\alpha}\right)\right) g^{(\alpha/2)}(1,s) ds. \qquad (3.13)
\end{aligned}$$

Now we estimate the first expression of (3.13).

Lemma 3.7. *Suppose $d \geqslant 2$ and $\alpha \in (1,2)$. Assume that D is a bounded $C^{1,1}$ domain. Then,*

$$\lim_{t\to 0} \frac{1}{t} \int_0^{t^{-2/\alpha}} \left(Q_D^{(2)}(s t^{2/\alpha}) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} s^{1/2} t^{1/\alpha}\right)\right) g^{(\alpha/2)}(1,s) ds$$

$$= \int_0^1 \left(Q_D^{(2)}(u) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} u^{1/2}\right)\right) \frac{\alpha}{2\Gamma(1-\frac{\alpha}{2})} u^{-1-\frac{\alpha}{2}} du.$$

Proof. By the change of the variables $u = t^{2/\alpha} s$ and the scaling property of $g^{(\alpha/2)}(t,x)$,

$$\begin{aligned}
&\frac{1}{t} \int_0^{t^{-2/\alpha}} \left(Q_D^{(2)}(s t^{2/\alpha}) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} s^{1/2} t^{1/\alpha}\right)\right) g^{(\alpha/2)}(1,s) ds \\
&= \int_0^1 \left(Q_D^{(2)}(u) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} u^{1/2}\right)\right) t^{-\frac{2}{\alpha}-1} g^{(\alpha/2)}(1, t^{-2/\alpha} u) du \\
&= \int_0^1 \left(Q_D^{(2)}(u) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} u^{1/2}\right)\right) \frac{g^{(\alpha/2)}(t,u)}{t} du. \qquad (3.14)
\end{aligned}$$

By (2.8), we have $g^{(\alpha/2)}(t,u)/(t) \leqslant c_1 u^{-1-\frac{\alpha}{2}}$. It follows from [**2**, Theorem 6.2] that there exists a constant c_2 such that

$$\left| Q_D^{(2)}(u) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} u^{\frac{1}{2}} \right) \right| \leqslant c_2 u, \quad u > 0.$$

Hence, the integrand in (3.14) is bounded above by $c_3 u^{-\frac{\alpha}{2}}$. Let $\varepsilon > 0$ and $\phi_\varepsilon \in C_b(\mathbb{R}^1)$ be such that $1_{B(0,\varepsilon)^c} \leqslant \phi_\varepsilon \leqslant 1_{B(0,\frac{\varepsilon}{2})^c}$. Hence, it follows from [**16**, Corollary 8.9] and the Lebesgue dominated convergence theorem that

$$\begin{aligned}
&\lim_{t\to 0} \int_0^1 \left(Q_D^{(2)}(u) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} u^{1/2} \right) \right) \phi_\varepsilon(u) \frac{g^{(\alpha/2)}(t,u)}{t} du \\
&= \int_0^1 \left(Q_D^{(2)}(u) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} u^{1/2} \right) \right) \phi_\varepsilon(u) \left(\lim_{t\to 0} \frac{g^{(\alpha/2)}(t,u)}{t} \right) du \\
&= \int_0^1 \left(Q_D^{(2)}(u) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} u^{1/2} \right) \right) \phi_\varepsilon(u) \frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)} u^{-1-\frac{\alpha}{2}} du.
\end{aligned}$$

Letting $\varepsilon \to 0$, we immediately get the assertion of the lemma by the Lebesgue dominated convergence theorem. □

The two lemmas below are about the second term in (3.13).

LEMMA 3.8. *Let $\alpha \in (1,2)$. Then, we have*

$$\lim_{t\to 0} \frac{1}{t} \int_{t^{-2/\alpha}}^\infty \left(|D| - Q_D^{(2)}(t^{2/\alpha} s) \right) g^{(\alpha/2)}(1,s) ds = \frac{1}{\Gamma\left(1-\frac{\alpha}{2}\right)} \int_D \mathbb{P}_x(\tau_D^{(2)} \leqslant 1) dx.$$

Proof. It follows from Fubini's theorem that

$$\begin{aligned}
&\int_{t^{-2/\alpha}}^\infty \left(|D| - Q_D^{(2)}(t^{2/\alpha} s) \right) g^{(\alpha/2)}(1,s) ds \\
&= \int_{t^{-2/\alpha}}^\infty \int_D \mathbb{P}_x\left(\tau_D^{(2)} \leqslant t^{2/\alpha} s \right) dx g^{(\alpha/2)}(1,s) ds = \int_D \int_{t^{-2/\alpha}}^\infty \mathbb{P}_x\left(\tau_D^{(2)} \leqslant t^{2/\alpha} s \right) g^{(\alpha/2)}(1,s) ds dx.
\end{aligned}$$

It follows from L'Hôpital's rule and (2.5) that

$$\begin{aligned}
&\lim_{t\to 0} \frac{\int_{t^{-2/\alpha}}^\infty \mathbb{P}_x(\tau_D^{(2)} \leqslant t^{2/\alpha} s) g^{(\alpha/2)}(1,s) ds}{t} \\
&= \lim_{t\to 0} \mathbb{P}_x\left(\tau_D^{(2)} \leqslant 1 \right) \frac{2}{\alpha} g^{(\alpha/2)}(1, t^{-2/\alpha}) t^{-\frac{2}{\alpha}-1} = \frac{1}{\Gamma\left(1-\frac{\alpha}{2}\right)} \mathbb{P}_x\left(\tau_D^{(2)} \leqslant 1 \right).
\end{aligned}$$

Now the result follows from the bounded convergence theorem. □

LEMMA 3.9. *Let $\alpha \in (1,2)$. Then,*

$$\lim_{t\to 0} \frac{1}{t^{1-\frac{1}{\alpha}}} \int_{t^{-2/\alpha}}^\infty s^{1/2} g^{(\alpha/2)}(1,s) ds = \frac{\alpha}{(\alpha-1)\Gamma\left(1-\frac{\alpha}{2}\right)}.$$

Proof. First note that by (2.9) we have $\lim_{t\to 0} \int_{t^{-2/\alpha}}^\infty s^{1/2} g^{(\alpha/2)}(1,s) ds = 0$. Hence, by L'Hôpital's rule and (2.5) we have

$$\lim_{t\to 0} \frac{1}{t^{1-\frac{1}{\alpha}}} \int_{t^{-2/\alpha}}^\infty s^{1/2} g^{(\alpha/2)}(1,s) ds = \frac{\alpha}{(\alpha-1)\Gamma\left(1-\frac{\alpha}{2}\right)}.$$ □

Proof of Theorem 1.2. The result follows from Lemmas 3.6, 3.7, 3.8, and 3.9. □

Here is a heuristic argument why the third term in the expansion of $\widetilde{Q}_D^{(\alpha)}(t)$ involves more than the mean curvature of D. When D is a bounded smooth domain, the following asymptotic expansion of $Q_D^{(2)}(t)$ is well known (for example see [17]):

$$Q_D^{(2)}(t) \sim \sum_{n=0}^{\infty} c_n t^{\frac{n}{2}}, \quad \text{as } t \to 0.$$

Hence, if the series indeed converges (there exists a case where this series does not converge) and if one could justify the interchange of the integral and the sum, one expects that

$$\lim_{t\to 0} \int_0^1 \left(Q_D^{(2)}(u) - \left(|D| - \frac{2|\partial D|}{\sqrt{\pi}} u^{1/2} \right) \right) \frac{g^{(\alpha/2)}(t,u)}{t} du = \int_0^1 \left(\sum_{n=2}^{\infty} c_n u^{n/2} \right) \lim_{t\to 0} \frac{g^{(\alpha/2)}(t,u)}{t} du$$

$$= \int_0^1 \left(\sum_{n=2}^{\infty} c_n u^{n/2} \right) \frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)} u^{-1-\frac{\alpha}{2}} du = \sum_{n=2}^{\infty} \frac{2c_n}{n-\alpha} \frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)}.$$

This suggests that even for smooth domains one cannot expect the third term in the expansion of $\widetilde{Q}_D^{(\alpha)}(t)$ to involve the mean curvature of D only. The limit contains the information for all the coefficients of the asymptotic expansion of $Q_D^{(2)}(t)$.

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