POTENTIAL THEORY OF SUBORDINATE BROWNIAN MOTIONS AND THEIR PERTURBATIONS

BY

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DISSERTATION

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Abstract

In this thesis, we study potential theoretic properties of harmonic functions and spectral problems of a large class of Lévy processes using probabilistic techniques.

In chapter 3 we prove sharp two-sided Green function estimates in bounded κ-fat domains D for a large class of Lévy processes, which can be considered as perturbations of certain subordinate Brownian motions. In particular, we prove that in bounded $C^{1,1}$ domains D, the Green function $G_D^Y(x,y)$ of symmetric Lévy processes Y whose Lévy densities are close to those of certain subordinate Brownian motions with characteristic exponent $\Psi(|\xi|) = \phi(|\xi|^2)$ satisfies

$$G_D^Y(x,y) \asymp \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right) \frac{1}{|x-y|^d\,\phi(|x-y|^{-2})}. \tag{0.0.1}$$

In chapter 4 we use the Green function comparability result to obtain a version of the boundary Harnack principle for positive harmonic functions that vanish outside a part of the boundary of D and some small ball with respect to perturbations of SBMs in bounded κ-fat domains D.

In chapter 5 we use the boundary Harnack principle to prove that the Martin boundary and the minimal Martin boundary of κ-fat domains D with respect to Y can be identified with the Euclidean boundary of D.

In chapter 6 we turn our attention to some spectral problems about relativistic stable processes. We establish the asymptotic expansion of the trace (partition function) $Z_D^m(t)$ of relativistic stable processes on bounded $C^{1,1}$ open sets and Lipschitz open sets as $t \to 0$.

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Chapter 1

Introduction

The study of fine potential theory of discontinuous (jump) Lévy processes started in the late 90's with the study of stable processes. The (symmetric) α-stable processes are Lévy processes with characteristic functions $e^{-t|\xi|^\alpha}$, where α is in $(0, 2]$. When $\alpha = 2$ one obtains Brownian motions, which have continuous sample paths; and when $0 < \alpha < 2$, the corresponding processes are pure jump processes. α-stable processes with $\alpha \in (0, 2)$ have various applications in physics, operation research, queuing theory, mathematical finance, and risk estimation. In physics, they are often called Lévy flights and are used in many concepts in physics such as turbulent diffusion, vortex dynamics, anomalous diffusion in rotating flows, and molecular spectral fluctuation. In mathematical finance, α-stable processes are used to model stock returns in incomplete markets.

Even though discontinuous stable processes are more suitable to model financial data than their continuous counterparts, it has been observed that the data tends to be more Gaussian in a large time scale, which can not be explained using stable processes. Relativistic stable processes with mass m are pure jump Lévy processes with characteristic functions $\exp(-t((m^{2/\alpha}+|\xi|^2)^{\alpha/2}-m)$ and seem to be good models to explain such cases. Relativistic stable processes also have applications in physics. When $\alpha = 1$ relativistic stable processes correspond to the kinetic energy of a relativistic particle with mass m.

Both stable and relativistic stable processes can be considered as members of a large class of Lévy processes called subordinate Brownian motions (SBM). Subordinate Brownian motions are Brownian motions observed at an independent random time (subordinator). When the Laplace exponent $\phi(\lambda)$ of the subordinator is given by $\phi(\lambda) = \lambda^{\alpha/2}$, the corresponding SBMs become α-stable processes and when $\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$, they correspond to relativistic α-stable processes with mass m. Hence, the family of SBMs contains a large class of interesting examples and is still more tractable than general Lévy processes. SBMs also arise naturally in finance and it is asserted that the asset price should be modeled as SBMs rather than Brownian motions.

The goal of this thesis is to study potential and spectral properties of SBMs and their perturbations. More precisely in chapter 3, we will prove the generalized 3G theorem for certain classes of SBMs and use this to prove that Green functions of SBMs and their perturbations are comparable in bounded κ-fat domains. In chapter 4, we use the Green function comparability result to prove the boundary Harnack principle (BHP) for perturbations of SBMs considered in chapter 3. In chapter 5, we also prove that the Martin boundary and the minimal Martin boundary of κ-fat domains with respect to perturbations of SBMs can be identified with the Euclidean boundary of the domain. In chapter 6, we turn our attention to some spectral problems of relativistic stable processes. We prove the asymptotic expansion of the trace (partition function) of relativistic stable processes for bounded $C^{1,1}$ open sets and bounded Lipschitz open sets.

The 3G theorem is a very important tool in studying (local) Schrödinger operators. It was established for Brownian motions in bounded Lipschitz domains for $d \geq 3$ in [28]. Later it was extended to bounded uniformly John domains for $d \geq 3$ in [2] (See [6, 33, 52, 56] for $d = 2$). For symmetric α-stable processes, $\alpha \in (0, 2)$, it was proved for bounded $C^{1,1}$ domains in [21, 22, 44]. More precisely, it was proved in [21, 22, 44] that for every $d > \alpha$ and any bounded $C^{1,1}$ domains D there exists a positive constant $c = c(D, \alpha)$ such that

$$\frac{\widetilde{G}_D(x, y)\widetilde{G}_D(y, z)}{\widetilde{G}_D(x, z)} \leq c \frac{|x - z|^{d-\alpha}}{|x - y|^{d-\alpha}|y - z|^{d-\alpha}}, \quad x, y, z \in D, \tag{1.0.1}$$

where $\widetilde{G}_D$ is the Green function of symmetric α-stable processes for D. Later (1.0.1) was extended to bounded Lipschitz domains for symmetric α-stable processes $(0 < \alpha < 2)$ in [34] and even to bounded κ-fat open sets in [54].

When the processes are discontinuous, there is a large class of additive functionals which are not continuous. Such additive functionals give rise to a large family of non-local Schrödinger operators. In order to deal with non-local Schrödinger operators, one needs the generalized 3G theorem, which gives an upper bound on $\widetilde{G}(x, y, z, w) := \widetilde{G}_D(x, y)\widetilde{G}_D(z, w)/\widetilde{G}_D(x, w)$ where y and z can be different (see Theorem 3.2.16). The generalized 3G theorem was proved in [35] for symmetric stable processes in bounded κ-fat open sets (see also [34]) and it can be stated as there exist constants

$c = c(D, \alpha)$ and $0 < \eta < \alpha$ such that for all $x, y, z, w \in D$

$$\widetilde{G}(x,y,z,w) \le c \left(\frac{|x-w| \wedge |y-z|}{|x-y|} \vee 1 \right)^{\eta} \left(\frac{|x-w| \wedge |y-z|}{|z-w|} \vee 1 \right)^{\eta} \frac{|x-w|^{d-\alpha}}{|x-y|^{d-\alpha}|z-w|^{d-\alpha}}. \quad (1.0.2)$$

We first extend (1.0.2) to subordinate Brownian motions considered in [40, 41, 43] in bounded κ-fat open sets D. Then we use the generalized 3G theorem to find concrete sufficient conditions for the Kato class of the subordinate Brownian motions (See Theorem 3.3.4, 3.3.5).

Sharp two-sided Green function estimates for a large class of subordinate Brownian motions X in κ-fat open sets D were established in [43]. The main goal of chapter 3 is to extend this result to more general Lévy processes. We prove that, for symmetric Lévy processes Y which can be considered as perturbations of processes X studied in [43], the Green function $G_D(x,y)$ of X in D and its counterpart $G_D^Y(x,y)$ are comparable for any bounded κ-fat domains D. Let J be the Lévy density of X, then the processes Y are symmetric purely discontinuous Lévy processes with the Lévy density J^Y such that $|\sigma(x)| \le c \max\{|x|^{-d+\rho}, 1\}$ for some constants $c > 0, \rho \in (0, d)$ where $\sigma(x) = J^Y(x) - J(x)$. Note that our main assumption is about the behavior of the Lévy density of Y near 0 and we do not impose any restriction about σ outside the unit ball other than σ being bounded there. The Lévy density of Y may vanish outside the unit ball. In this case Y only have jumps of size less than 1 and they correspond to a natural generalization of truncated stable processes studied in [37, 38]. One of the main tools used in this chapter is the drift transform studied in [26]. We first use the drift transform and our generalized 3G theorem to show that, under the additional assumption that $J^Y(x) \ge J(x)$ for all $x \in \mathbb{R}^d$, $G_D^Y(x,y)$ is comparable to $G_D(x,y)$ for any bounded κ-fat (not necessarily connected) open sets D (Theorem 3.4.6). Then we deal with the general case where σ can take both signs (Theorem 3.4.13).

The boundary Harnack principle (BHP) for classical nonnegative harmonic functions is a very deep result in potential theory and has very important applications in probability and potential theory. The boundary Harmonic principle for nonnegative harmonic functions with respect to non-local operators (or, equivalently, discontinuous Markov processes) was first established in [10] for symmetric stable processes in Lipschitz domains. Since then, the result has been generalized in various directions. In one direction, the BHP is established for more general open sets than Lipschitz domains. In [54], the boundary Harnack principle was established for κ-fat open sets for nonnegative harmonic functions with respect to rotationally symmetric stable processes. The boundary Harnack principle has also been established for arbitrary open sets. In [12], the authors proved the boundary Harnack principle for rotationally symmetric stable processes in arbitrary open sets with the constant not depending on the geometry of the open sets. This type of result is known as the uniform boundary Harnack principle. In another direction, the boundary Harnack principle has been established for nonnegative harmonic functions with respect to different classes of Lévy processes. In [40], the authors proved the boundary Harnack principle for nonnegative harmonic functions with respect to a wide class of subordinate Brownian motions in bounded κ-fat open sets. In [42], the authors proved the uniform boundary Harnack principle for very general Lévy processes which are not necessarily subordinate Brownian motions. In [37], the boundary Harnack principle was proved for nonnegative harmonic functions with respect to truncated stable processes in κ-fat open sets with an extra condition that the harmonic functions vanish outside a small ball as well as near a part of the boundary of the domain and for nonnegative harmonic functions with respect to truncated stable processes in bounded convex domains without the extra condition mentioned above.

In chapter 4, we generalize the boundary Harnack principle for symmetric Lévy processes Y, which can be considered as perturbations of subordinate Brownian motions that appeared in chapter 3. In this version of the boundary Harnack principle, we assume that the harmonic functions vanish outside a small ball as well as a part of the boundary of the domain. This is not a merely technical point since it was proved in [37] that without the condition that harmonic functions vanishing outside a small ball, the boundary Harnack principle for truncated stable processes fails to hold in non-convex domains. One of the main ingredients to prove the boundary Harnack principle is the uniform Green functions comparability result of the Green functions $G_D^X(x,y)$ and $G_D^Y(x,y)$ of subordinate Brownian motion X and their perturbations Y for all sufficiently small bounded κ-fat domains D.

In chapter 5, we study the Martin boundary and the minimal Martin boundary of bounded κ-fat domains with respect to Y. Superharmonic and harmonic functions with respect to killed Markov processes have been studied in the context of general theory of Markov processes and their potential theory in [46]. However, it was not until late 1990's and early 2000's that special case of harmonic functions with respect to killed stable processes was investigated in [11, 23, 47]. Later the study of the Martin boundary of various jump processes has been investigated in [16, 17, 37, 38, 40]. In this chapter, we show that for bounded κ-fat domains, the Martin boundary and the minimal Martin boundary with respect to perturbations of subordinate Brownian motions can be identified with the Euclidean boundary of the domain. Furthermore when the domain is a bounded $C^{1,1}$ domain, we get sharp two-sided estimates for the Martin kernel of the domain with respect to Y. The (uniform) Green functions comparability result plays a crucial role in this chapter.

In chapter 6, we study the spectral problems of relativistic stable processes. In particular, we study the asymptotic behavior of the trace of the (killed) relativistic stable processes as $t \to 0$. The asymptotic behaviors of the trace $Z_D(t)$ of killed Brownian motions (i.e., killed symmetric α-stable processes with $\alpha = 2$) in bounded domains D of $\mathbb{R}^d$ have been extensively studied by many authors. It is shown in [7] that, when D is a bounded $C^{1,1}$ domain,

$$\left| Z_D(t) - (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| \right) \right| \le \frac{c|D| t^{1-d/2}}{R^2}, \quad t > 0.$$

The following asymptotic result

$$Z_D(t) = (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| + o(t^{1/2}) \right), \quad t \to 0, \tag{1.0.3}$$

was proved in [13] when D is a bounded C^1 domain. (1.0.3) was subsequently extended to Lipschitz domains in [14].

The asymptotic behaviors of the trace $Z_D^0(t)$ of killed symmetric α-stable processes, $0 < \alpha < 2$, in open sets of $\mathbb{R}^d$ have been studied in [3, 4]. It was shown in [3] that, for any bounded $C^{1,1}$ open sets D,

$$\left| Z_D^0(t) - \frac{C_1 |D|}{t^{d/\alpha}} + \frac{C_2 |\partial D| t^{1/\alpha}}{t^{d/\alpha}} \right| \le \frac{c|D| t^{2/\alpha}}{r_0^2 t^{d/\alpha}},$$

where C_1 and C_2 are the same as in Theorem 6.1.1 and c is a positive constant depending on d and α only. It was shown in [4] that, when D is a bounded Lipschitz domain, $Z_D^0(t)$ satisfies

$$t^{d/\alpha} Z_D^0(t) = C_1 |D| - C_2 \mathcal{H}^{d-1}(\partial D) t^{1/\alpha} + o(t^{1/\alpha}).$$

In chapter 6, we prove the asymptotic expansion of the trace (partition functions) $Z_D^m(t)$ of relativistic stable processes as $t \to 0$ in Theorem 6.1.1 and 6.1.2 for bounded $C^{1,1}$ open sets and bounded Lipschitz open sets, respectively. For relativistic stable processes, the corresponding trace $Z_D^m(t)$ has similar first and second leading terms as the trace of stable processes but there appear extra twist terms. We note here that as in the case of stable processes, the first leading term of $Z_D^m(t)$ involves an area of the domain and the second leading term involves a perimeter of the domain.

In this thesis we always assume that $\alpha \in (0, 2)$ and d is a positive integer with $d > \alpha$. We will use the following convention: The values of the constants C_0, C_1, M, $r_0, r_1, r_2, \cdots$ and ε_1 will remain the same throughout this thesis, while $c, c_1, c_2, \cdots$ stand for constants whose values are unimportant and which may change from one appearance to another. The labeling of the constants $c_0, c_1, c_2, \cdots$ starts anew in the statement of each result. We use ":=" to denote a definition, which is read as "is defined to be". We denote $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$. $f(t) \asymp g(t)$, $t \to 0$ ($f(t) \asymp g(t)$, $t \to \infty$, respectively) means that the quotient $f(t)/g(t)$ stays bounded between two positive constants as $t \to 0$ (as $t \to \infty$, respectively). For any open set U, we denote by $\delta_U(x)$ the distance of a point x to the boundary of U, i.e., $\delta_U(x) = \text{dist}(x, \partial U)$.

Chapter 2

Preliminaries

2.1 Lévy Processes

Definition 2.1.1. *Stochastic processes X_t are called Lévy processes if they satisfy*

1. For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.

2. $X_0 = 0$ almost surely.

3. The distribution of $X_{t+s} - X_s$ does not depend on s.

4. There is $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}[\Omega_0] = 1$ such that, for every $\omega \in \Omega_0$, $X_t(\omega)$ is right continuous in $t \geq 0$ and has left limits in $t > 0$.

For any probability measure μ, we define its characteristic function $\hat{\mu}(z) := \int e^{i<z,\xi>} f(\xi) d\xi$. A measure μ is said to be infinitely divisible if for any $n \geq 1$ there exists a measure μ_n such that $\mu = \mu_n^{*n}$, where $*$ is a convolution of measure. It is well known that ([50, Theorem 7.10] stochastic processes X are Lévy processes if and only if μ is infinitely divisible, where $\mu = \mathbb{P}_{X_1}$. It is also well known that if μ is infinitely divisible, its characteristic function $\hat{\mu}$ has a unique decomposition called the Lévy-Khintchine formula.

Theorem 2.1.2. *1. If μ is an infinitely divisible distribution on $\mathbb{R}^d$, then*

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2} < z, Az > + i < \gamma, z > + \int_{\mathbb{R}^d} \left(e^{i<z,x>} - 1 - i < z, x > 1_{B(0,1)}(x) \right) \nu(dx) \right], \quad z \in \mathbb{R}^d, \tag{2.1.1}$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} \left(|x|^2 \wedge 1 \right) \nu(dx) < \infty, \tag{2.1.2}$$

and $\gamma \in \mathbb{R}^d$.

2. The representation of $\hat{\mu}(z)$ *in* (2.1.1) *by* A, ν, *and* γ *is unique.*

3. Conversely, if A *a symmetric nonnegative-definite* $d \times d$ *matrix,* ν *is a measure satisfying* (2.1.2), *and* $\gamma \in \mathbb{R}^d$, *then there exists an infinitely divisible distribution* μ *whose characteristic function is given by* (2.1.1).

Example 2.1.3. *1. Let* $A = 1$, $\gamma = 0$, *and* $\nu \equiv 0$ *in* (2.1.1). *The corresponding processes* X *are Brownian motions, whose sample paths are continuous almost surely.*

2. Let $A \equiv 0$, $\gamma = 0$, *and* $\nu(dx) = \frac{c(d,\alpha)}{|x|^{d+\alpha}}dx$ *in* (2.1.1), *where* $\alpha \in (0, 2)$. *The corresponding processes are* α*-stable processes of index* α. *Unlike Brownian motions, the sample path of* α*-stable processes is discontinuous.*

2.2 Subordinate Brownian Motions

The purpose of this section is to develop the theory of subordinate Brownian motions under the assumption that the Laplace exponent of the subordinator is a complete Bernstein function and is comparable to a regularly varying function at infinity. We will closely follow the argument and notations in [41]. We first recall the definition of subordinators and their relation to Bernstein functions. Recall that a subordinator $S = (S_t)$ is an increasing Lévy process taking values in $[0, \infty)$ with $S_0 = 0$. A subordinator is completely characterized by its Laplace exponent ϕ via

$$\mathbb{E}\left[\exp(-\lambda S_t)\right] = \exp(-t\phi(\lambda)), \quad \lambda > 0.$$

The Laplace exponent ϕ can be written as

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\mu(dt),$$

where $b \geq 0$ and μ is a σ-finite measure on $(0, \infty)$ satisfying

$$\int_0^\infty (1 \wedge t)\, \mu(dt) < \infty.$$

The constant b is called the drift, and μ the Lévy measure of the subordinator S. Now we recall the definition of Bernstein functions and the relation between Bernstein functions and subordinators. A C^∞ function $\phi : (0,\infty) \to [0,\infty)$ is called a Bernstein function if $(-1)^n D^n \phi \le 0$ for every positive integers n. Every Bernstein function has a representation (cf. [51, Theorem 3.2])

$$\phi(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\mu(dt), \tag{2.2.1}$$

where $a, b \ge 0$ and μ is a measure on $(0,\infty)$ satisfying $\int_0^\infty (1 \wedge t)\, \mu(dt) < \infty$. Thus a nonnegative function ϕ on $(0,\infty)$ is the Laplace exponent of a subordinator if and only if it is a Bernstein function with $\phi(0+) = 0$. A Bernstein function is called a complete Bernstein function if the Lévy measure μ has a completely monotone density $\mu(t)$, i.e., $(-1)^n D^n \mu \ge 0$ for every non-negative integer n.

The potential measure of the subordinator S is defined by

$$U(A) = \mathbb{E} \int_0^\infty 1_{\{S_t \in A\}} dt, \quad A \subset [0,\infty).$$

Note that $U(A)$ is the expected time that the subordinator S spends in the set A. The Laplace transform of the measure U is given by

$$\mathcal{L}U(\lambda) = \int_0^\infty e^{-\lambda t} dU(t) = \mathbb{E} \int_0^\infty \exp(-tS_t) dt = \frac{1}{\phi(\lambda)}.$$

From now on we will impose the condition that every Bernstein function that appears as the Laplace exponent of some subordinator is always a complete Bernstein function with $\mu(0,\infty) = \infty$ and zero drift $b = 0$. That is,

$$\phi \textit{ is a complete Bernstein function, } \mu(0,\infty) = \infty, \textit{ and } b = 0. \tag{2.2.2}$$

Note that when $\mu(0,\infty) = \infty$ and ϕ is a completely Bernstein function, the potential measure U of S has a completely monotone density u. (See [41, Corollary 2.3].) Next we will impose another condition on ϕ to determine the asymptotic behavior of u and μ near the origin. Note that this essentially follows from Tauberian type theorems and the monotone density theorem by using the information about ϕ near infinity. If we impose a condition about ϕ near 0, then it is possible to determine the asymptotic behavior of u and μ near infinity. Since we will be dealing with processes on bounded sets D, the behavior near the origin is more important and we will just impose the condition about ϕ near infinity.

From now on, we will assume that there exist $\alpha \in (0,2)$ and a function $\ell : (0,\infty) \to (0,\infty)$ which is measurable, locally bounded and slowly varying at infinity such that

$$\phi(\lambda) \asymp \lambda^{\alpha/2}\ell(\lambda), \quad \lambda \to \infty. \tag{2.2.3}$$

Under conditions (2.2.2) and (2.2.3), it follows from [41, Theorem 2.9, 2.10] that

Theorem 2.2.1 (Theorem 2.9 [41])**.** *Let S be a complete subordinator with Laplace exponent ϕ satisfying* (2.2.3)*. Then the potential density u of S satisfies*

$$u(t) \asymp t^{-1}\phi(t^{-1})^{-1} \asymp \frac{t^{\alpha/2-1}}{\ell(t^{-1})}, \quad t \to 0+\,.$$

Theorem 2.2.2 (Theorem 2.10 [41])**.** *Let S be a complete subordinator with Laplace exponent ϕ satisfying* (2.2.3)*. Then the Lévy density μ of S satisfies*

$$\mu(t) \asymp t^{-1}\phi(t^{-1}) \asymp t^{-\alpha/2-1}\ell(t^{-1}), \quad t \to 0+\,.$$

Now we will define and investigate subordinate Brownian motions (SBM). Loosely speaking, subordinate Brownian motions are just time changed Brownian motions. The time is represented by an independent increasing Lévy process (subordinator) and can be considered as an operational time or an intrinsic time. When we deal with stochastic models, it is often desirable to use subordinate Brownian motions rather than Brownian motions and this is one motivation for the study of subordinate Brownian motions. Now we will define SBMs rigorously. Let $B = (B_t, \mathbb{P}_x)$ be Brownian motions in $\mathbb{R}^d$ with transition density $p(t,x,y) = p(t, y-x)$ given by

$$p(t,x) = (4\pi t)^{-d/2}\exp(-\frac{|x|^2}{4t}), \quad t > 0, x, y \in \mathbb{R}^d.$$

Note that Brownian motions here are twice faster than usual Brownian motions in the literature.

The semigroup $(P_t : t \geq 0)$ of B is defined by $P_t f(x) = \mathbb{E}_x[f(B_t)] = \int_{\mathbb{R}^d} p(t,x,y) f(y) dy$, where f is a nonnegative Borel function on $\mathbb{R}^d$. Note that when $d \geq 3$, the Green function $G^2(x,y) = G^2(x-y), x, y \in \mathbb{R}^d$ of B exists and

$$G^2(x) = \int_0^\infty p(t,x) dt = \frac{\Gamma(d/2-1)}{4\pi^{d/2}} |x|^{-d+2}.$$

Let $S = (S_t, t \geq 0)$ be a complete subordinator which is independent to B with the Laplace exponent $\phi(\lambda)$, the Lévy measure μ and the potential measure U. We will always assume that ϕ satisfies (2.2.2) and (2.2.3). Stochastic processes defined by $X_t := B_{S_t}$ are called subordinate Brownian motions and they are Lévy processes with the Lévy exponent $\Phi(x) = \phi(|x|^2)$ (see [50, pp. 197-198]). The semigroup of X is given by $(Q_t : t \geq 0)$

$$Q_t f(x) = \mathbb{E}_x[f(X_t)] = \mathbb{E}_x[f(B(S_t))] = \int_0^\infty P_s f(x) \mathbb{P}(S_t \in ds). \tag{2.2.4}$$

From (2.2.4), it is easy to see that the transition density (heat kernel) of X is given by $q(t,x,y) = q(t, x-y)$, where

$$q(t,x) = \int_0^\infty p(s,x) \mathbb{P}(S_t \in ds).$$

We will always assume that the Lévy processes X are transient. According to the criterion of Chung-Fuchs type (see [50, pp. 252-253]), the Lévy processes X are transient if and only if for some small $r > 0$, $\int_{\{|x|<r\}} \frac{1}{\Phi(x)} dx < \infty$. Since $\Phi(x) = \phi(|x|^2)$, it follows that the Lévy processes X are transient if and only if

$$\int_{0+} \frac{\lambda^{d/2-1}}{\phi(\lambda)} d\lambda < \infty. \tag{2.2.5}$$

This is always true when $d \geq 3$ and when $d = 1, 2$ we will impose another assumption on ϕ to guarantee that the Lévy processes X are transient. This condition is as follows. For $d \leq 2$, there exists $\gamma \in [0, d/2)$ such that

$$\liminf_{\lambda \to 0} \frac{\phi(\lambda)}{\lambda^\gamma} > 0. \tag{2.2.6}$$

Note that when (2.2.6) is true, (2.2.5) is also true. Now we will define the potential measure and the Green function of X. For $x \in \mathbb{R}^d$ and a Borel subset $A \in \mathbb{R}^d$, the potential measure of X is given by

$$\begin{aligned} G(x,A) &= \mathbb{E}_x \int_0^\infty 1_{\{X_t \in A\}} dt = \int_0^\infty Q_t 1_A(x) dt = \int_0^\infty \int_0^\infty P_s 1_A(x) \mathbb{P}(S_t \in ds) dt \\ &= \int_0^\infty P_s 1_A u(s) ds = \int_A \int_0^\infty p(s,x,y) ds dy. \end{aligned}$$

Let $G(x,y)$ denote the density of the potential measure $G(x,\cdot)$. Then it is easy to see that $G(x,y) = G(y-x)$, where

$$G(x) = \int_0^\infty p(t,x) U(dt) = \int_0^\infty p(t,x) u(t) dt.$$

The Lévy measure Π of X is given by

$$\Pi(A) = \int_A \int_0^\infty p(t,x) \mu(dt) dx = \int_A J(x) dx, \quad A \subset \mathbb{R}^d,$$

where

$$J(x) := \int_0^\infty p(t,x) \mu(dt) = \int_0^\infty p(t,x) \mu(t) dt \tag{2.2.7}$$

is the Lévy density of X. Define the function $j : (0,\infty) \to (0,\infty)$ by

$$j(r) = \int_0^\infty (4\pi)^{-d/2} t^{-d/2} \exp(-\frac{r^2}{4t}) \mu(dt), \quad r > 0.$$

Note that by (2.2.7), $J(x) = j(|x|)$, $x \in \mathbb{R}^d \setminus \{0\}$. Since $x \to p(t,x)$ is continuous and radially decreasing, we conclude that both G and J are continuous on $\mathbb{R}^d$ and radially decreasing.

Chapter 3

Green Function Estimations

3.1 Preliminaries

In this section, we recall some preliminary results about subordinate Brownian motions considered in [40, 41]. The following theorem establishes the asymptotic behaviors of G and j near the origin (see [43, Theorem 2.9, 2.11]).

Theorem 3.1.1.

(i)

$$G(x) \asymp \frac{1}{|x|^d \phi(|x|^{-2})} \asymp \frac{1}{|x|^{d-\alpha} \ell(|x|^{-2})}, \qquad |x| \to 0.$$

(ii)

$$J(x) = j(|x|) \asymp \frac{\phi(|x|^{-2})}{|x|^d} \asymp \frac{\ell(|x|^{-2})}{|x|^{d+\alpha}}, \qquad |x| \to 0.$$

For any open set D, we use τ_D to denote the first exit time of D, i.e., $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Given an open set $D \subset \mathbb{R}^d$, we define $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and $X_t^D(\omega) = \partial$ if $t \geq \tau_D(\omega)$, where ∂ is a cemetery state. X^D is called the killed subordinate Brownian motion X in D. We now recall the definition of harmonic functions with respect to X.

Definition 3.1.2. *Let D be an open subset of $\mathbb{R}^d$. A nonnegative function u defined on $\mathbb{R}^d$ is said to be*

(1) *harmonic in D with respect to X if*

$$u(x) = \mathbb{E}_x \left[u(X_{\tau_B}) \right], \qquad x \in B$$

for every open set B whose closure is a compact subset of D;

(2) *regular harmonic in D with respect to X if for each $x \in D$,*

$$u(x) = \mathbb{E}_x \left[u(X_{\tau_D}) \right].$$

The following version of the Harnack inequality is [43, Theorem 2.14].

Theorem 3.1.3. *For any $L > 0$, there exists a positive constant $c = c(d, \phi, L) > 0$ such that the following is true: If $x_1, x_2 \in \mathbb{R}^d$ and $r \in (0, 1)$ are such that $|x_1 - x_2| < Lr$, then for every nonnegative function u which is harmonic with respect to X in $B(x_1, r) \cup B(x_2, r)$, we have*

$$c^{-1} u(x_2) \le u(x_1) \le c u(x_2).$$

For any open set D in $\mathbb{R}^d$, we will use $G_D(x, y)$ to denote the Green function of X^D. Using the continuity and the radial decreasing property of G, we can easily check that G_D is continuous in $(D \times D) \setminus \{(x, x) : x \in D\}$. We will frequently use the well-known fact that $G_D(\cdot, y)$ is harmonic in $D \setminus \{y\}$, and regular harmonic in $D \setminus \overline{B(y, \varepsilon)}$ for every $\varepsilon > 0$.

The following concept was introduced in [54].

Definition 3.1.4. *Let $\kappa \in (0, 1/2]$. We say that an open set D in $\mathbb{R}^d$ is κ-fat if there exists $r_0 > 0$ such that for each $Q \in \partial D$ and $r \in (0, r_0)$, $D \cap B(Q, r)$ contains a ball $B(A_r(Q), \kappa r)$. The pair (r_0, κ) is called the characteristics of the κ-fat open set D.*

The following boundary Harnack principle is [41, Theorem 4.22].

Theorem 3.1.5. ([40, Theorem 4.8], [41, Theorem 4.22]) Suppose that D is a κ-fat open set with characteristics (r_0, κ). There exists a constant $c = c(d, r_0, \kappa, \phi) > 1$ such that, if $r \in (0, r_0 \wedge \frac{1}{4}]$ and $Q \in \partial D$, then for any nonnegative functions u, v in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(Q, 2r)$ with respect to X and vanish in $D^c \cap B(Q, 2r)$, we have

$$c^{-1} \frac{u(A_r(Q))}{v(A_r(Q))} \le \frac{u(x)}{v(x)} \le c \frac{u(A_r(Q))}{v(A_r(Q))}, \qquad x \in D \cap B(Q, \frac{r}{2}).$$

3.2 Generalized 3G Theorem

In this section, we prove a generalized 3G theorem for X in a bounded κ-fat open set D. This theorem will play an important role later in this chapter.

We first present some preliminary results which are valid for any bounded open set D. The following proposition is a combination of [43, Proposition 3.2 and Lemma 3.3].

Proposition 3.2.1. *Suppose D is a bounded open set in $\mathbb{R}^d$. (i) There exists a positive constant $C_0 = C_0(\mathrm{diam}(D), \phi, d)$ such that*

$$G_D(x,y) \le C_0 \frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})}, \qquad x,y \in D. \tag{3.2.1}$$

(ii) For every $L > 0$, there exists $c = c(\mathrm{diam}(D), \phi, L, d) > 0$ such that for every $|x-y| \le L(\delta_D(x) \wedge \delta_D(y))$,

$$G_D(x,y) \ge c \frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})}.$$

In the remainder of this section, we assume D is a bounded κ-fat open set with characteristics (r_0, κ). Without loss of generality we may assume that $r_0 \le 1/4$. Recall that for each $Q \in \partial D$ and $r \in (0, r_0)$, $A_r(Q)$ is a point in $D \cap B(Q,r)$ satisfying $B(A_r(Q), \kappa r) \subset D \cap B(Q,r)$. Since $G_D(z,\cdot)$ is regular harmonic in $D \setminus \overline{B(z,\varepsilon)}$ for every $\varepsilon > 0$ and vanishes outside D, the following result follows easily from Theorem 3.1.5.

Theorem 3.2.2. *There exists a constant $c = c(d, r_0, \kappa, \phi) > 1$ such that for any $Q \in \partial D$, $r \in (0, r_0]$ and $z, w \in D \setminus B(Q, 2r)$, we have*

$$c^{-1}\frac{G_D(z, A_r(Q))}{G_D(w, A_r(Q))} \le \frac{G_D(z,x)}{G_D(w,x)} \le c\,\frac{G_D(z, A_r(Q))}{G_D(w, A_r(Q))}, \quad x \in D \cap B\left(Q, \frac{r}{2}\right).$$

Using the uniform convergence theorem ([8, Theorem 1.2.1]), we can choose $r_1 \le r_0$ such that if $r \le r_1$ then

$$\frac{1}{2} \le \min_{\frac{1}{6} \le \lambda \le 2\kappa^{-1}} \frac{\ell((\lambda r)^{-2})}{\ell(r^{-2})} \le \max_{\frac{1}{6} \le \lambda \le 2\kappa^{-1}} \frac{\ell((\lambda r)^{-2})}{\ell(r^{-2})} \le 2. \tag{3.2.2}$$

Fix $z_0 \in D$ with $\kappa r_1 < \delta_D(z_0) < r_1$ and let $\varepsilon_1 := \kappa r_1/24$. For $x, y \in D$, we define $r(x,y) :=$

$\delta_D(x) \vee \delta_D(y) \vee |x-y|$ and

$$\mathcal{B}(x,y) := \begin{cases} \left\{A \in D:\ \delta_D(A) > \frac{\kappa}{2} r(x,y),\ |x-A| \vee |y-A| < 5r(x,y)\right\} & \text{if } r(x,y) < \varepsilon_1 \\ \{z_0\} & \text{if } r(x,y) \geq \varepsilon_1. \end{cases}$$

Note that if $r(x,y) < \varepsilon_1$

$$\frac{1}{6}\delta_D(A) \ \leq\ \delta_D(x) \vee \delta_D(y) \vee |x-y| \ \leq\ 2\kappa^{-1}\delta_D(A), \qquad A \in \mathcal{B}(x,y).$$

Thus by (3.2.2), if $r(x,y) < \varepsilon_1$,

$$\frac{1}{2} \ \leq\ \frac{\ell((\delta_D(A))^{-2})}{\ell((r(x,y))^{-2})} \ \leq\ 2, \qquad A \in \mathcal{B}(x,y).$$

Let

$$C_1 := C_0 2^{d-\alpha} \delta_D(z_0)^{-d+\alpha} \cdot \sup_{\delta_D(z_0)/2 \leq r \leq \mathrm{diam}(D)} \ell(r^{-2})^{-1}$$

so that, by Proposition 3.2.1(i), $G_D(\cdot, z_0)$ is bounded from above by C_1 on $D \setminus B(z_0, \delta_D(z_0)/2)$. Now we define

$$g(x) := G_D(x, z_0) \wedge C_1.$$

Note that if $\delta_D(z) \leq 6\varepsilon_1$, then $|z - z_0| \geq \delta_D(z_0) - 6\varepsilon_1 \geq \delta_D(z_0)/2$ since $6\varepsilon_1 < \delta_D(z_0)/4$, and therefore $g(z) = G_D(z, z_0)$.

The following result is established in [43].

Theorem 3.2.3 ([43, Theorem 1.2])**.** *There exists $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ such that for every $x, y \in D$*

$$c^{-1}\, \frac{g(x)g(y)}{g(A)^2 |x-y|^d \phi(|x-y|^{-2})} \ \leq\ G_D(x,y) \ \leq\ c\, \frac{g(x)g(y)}{g(A)^2 |x-y|^d \phi(|x-y|^{-2})}, \quad A \in \mathcal{B}(x,y).$$

Lemma 3.2.4. *There exist positive constants $c = c(d, r_0, \kappa, \phi)$, $\beta = \beta(d, r_0, \kappa, \phi) < \alpha$ and $r_2 \in (0, r_1]$ such that for any $Q \in \partial D$, $r \in (0, r_2)$, and nonnegative function u on $\mathbb{R}^d$ which is harmonic* *with respect to X in $D \cap B(Q,r)$ we have*

$$u(A_r(Q)) \,\le\, c\, \left(\frac{r}{s}\right)^{\beta} \frac{\ell(s^{-2})}{\ell(r^{-2})}\, u(A_s(Q)), \qquad s \in (0,r).$$

Proof. Without loss of generality, we assume $Q = 0$. Let $a_k := \left(\frac{\kappa}{2}\right)^k$ for $k = 0, 1, \cdots$. By using [41, Proposition 4.10] instead of [40, Proposition 3.8] and repeating the proof of [40, Lemma 5.2], we easily see that [40, Lemma 5.2] is valid in the present case. Thus there exist positive constants $c = c(d, r_0, \kappa, \phi)$, $\beta = \beta(d, r_0, \kappa, \phi) < \alpha$, and $R_1 \in (0, r_1]$ such that for every $k = 0, 1, \cdots$,

$$u(A_r(0)) \,\le\, c_1 \left(\frac{r}{a_k r}\right)^{\beta} \frac{\ell((a_k r)^{-2})}{\ell(r^{-2})}\, u(A_{a_k r}(0)), \qquad r \in (0, R_1].$$

Since ℓ is slowly varying at ∞, there exist $R_2 = R_2(d,\beta,\ell) \in (0, R_1]$ and $c_2 = c_2(d,\beta,\ell) > 0$ such that

$$\frac{s^{\beta}}{\ell(s^{-2})} \,\le\, c_2\, \frac{r^{\beta}}{\ell(r^{-2})}, \qquad \forall\, 0 < s < r \le R_2. \tag{3.2.3}$$

Thus if $r \le R_2$ and $a_{k+1} r < s \le a_k r$, by (3.2.3) and Theorem 3.1.3,

$$u(A_r(0)) \,\le\, c_3\, \frac{r^{\beta}}{\ell(r^{-2})} \frac{\ell((a_k r)^{-2})}{(a_k r)^{\beta}}\, u(A_s(0)) \,\le\, c_4\, \frac{r^{\beta}}{\ell(r^{-2})} \frac{\ell(s^{-2})}{s^{\beta}}\, u(A_s(0))$$

for some positive constants c_3, c_4 independent of s. □

Applying [41, Lemma 4.19] to Green functions, we have the following.

Lemma 3.2.5 (Carleson's estimate)**.** *There exists $c = c(d, r_0, \kappa, \phi) > 1$ such that for every $Q \in \partial D$, $r \in (0, 1/4)$, and $y \in D \setminus \overline{B(Q, 4r)}$*

$$G_D(x,y) \,\le\, c\, G_D(A_r(Q), y), \quad x \in D \cap B(Q,r). \tag{3.2.4}$$

For every $x, y \in D$, let Q_x and Q_y be points on ∂D such that $\delta_D(x) = |x - Q_x|$ and $\delta_D(y) = |y - Q_y|$ respectively. It is easy to check that if $r(x,y) < \varepsilon_1$, $A_{r(x,y)}(Q_x)$, $A_{r(x,y)}(Q_y) \in \mathcal{B}(x,y)$. (For example, see [35, page 123].) Moreover, since $g(A_1) \asymp g(A_2)$ for all $A_1, A_2 \in \mathcal{B}(x,y)$ by Theorem

3.1.3, we have in particular

$$g(A_{r(x,y)}(Q_x)) \asymp g(A_{r(x,y)}(Q_y)) \asymp g(A_{x,y}) \quad \text{for all } A_{x,y} \in \mathcal{B}(x,y). \tag{3.2.5}$$

This simple but useful fact will be used later in this section.

Using our Theorem 3.1.3 and Lemma 3.2.5, the proofs of the next four lemmas are the same as those of [35, Lemmas 3.8–3.11], so we omit the proofs.

Lemma 3.2.6. *There exists $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ such that for every $x, y \in D$ with $r(x,y) < \varepsilon_1$,*

$$g(z) \le c\, g(A_{r(x,y)}(Q_x)), \quad z \in D \cap B(Q_x, r(x,y)). \tag{3.2.6}$$

Lemma 3.2.7. *There exists $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ such that for every $x, y \in D$*

$$g(x) \vee g(y) \le c\, g(A), \quad A \in \mathcal{B}(x,y).$$

Lemma 3.2.8. *If $x, y, z \in D$ satisfy $r(x,z) \le r(y,z)$, then there exists $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ such that*

$$g(A_{x,y}) \le c\, g(A_{y,z}) \qquad \text{for every } (A_{x,y}, A_{y,z}) \in \mathcal{B}(x,y) \times \mathcal{B}(y,z).$$

Lemma 3.2.9. *There exists $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ such that for every $x, y, z, w \in D$ and $(A_{x,y}, A_{y,z}, A_{z,w}, A_{x,w}) \in \mathcal{B}(x,y) \times \mathcal{B}(y,z) \times \mathcal{B}(z,w) \times \mathcal{B}(x,w)$,*

$$g(A_{x,w})^2 \le c \left(g(A_{x,y})^2 + g(A_{y,z})^2 + g(A_{z,w})^2 \right).$$

Combining Theorem 3.2.3, Lemmas 3.2.7 and 3.2.8, and applying Theorem 3.1.1(i), we have the following 3G Theorem.

Theorem 3.2.10 (3G theorem)**.** *There exists $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ such that for every $x, y, z \in D$*

$$\frac{G_D(x,y) G_D(y,z)}{G_D(x,z)} \le c \frac{G(x,y) G(y,z)}{G(x,z)} \asymp \frac{\phi(|x-z|^{-2})}{\phi(|x-y|^{-2}) \phi(|y-z|^{-2})} \frac{|x-z|^d}{|x-y|^d |y-z|^d}.$$

In the remainder of this thesis, β will always stand for the constant from Lemma 3.2.4.

Lemma 3.2.11. *There exists* $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ *such that for every* $x, y \in D$ *with* $r(x,y) < \varepsilon_1$,

$$g(A_{x,y}) \geq c \frac{r(x,y)^\beta}{\ell((r(x,y))^{-2})}, \quad \textit{for all } A_{x,y} \in \mathcal{B}(x,y).$$

Proof. Let $A := A_{r(x,y)}(Q_x)$. Note that $g(\cdot) = G_D(\cdot, z_0)$ is harmonic in $D \cap B(Q_x, 2\varepsilon_1)$. Since $r(x,y) < \varepsilon_1$, by Lemma 3.2.4 (recall $\varepsilon_1 = \kappa r_1/24$),

$$g(A) = G_D(A, z_0) \geq c \left(\frac{r(x,y)}{2\varepsilon_1}\right)^\beta \frac{\ell((2\varepsilon_1)^{-2}}{\ell((r(x,y))^{-2})} G_D(A_{2\varepsilon_1}(Q_x), z_0).$$

Note that $\delta_D(z_0) \geq r_1\kappa = 24\varepsilon_1$ and $\delta_D(A_{2\varepsilon_1}(Q_x)) > 2\kappa\varepsilon_1$. Thus by Proposition 3.2.1(ii) we have $G_D(A_{2\varepsilon_1}(Q_x), z_0) > c_1 > 0$. This completes the proof of (3.2.5). □

Lemma 3.2.12. *There exists* $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ *such that for every* $x, y, z \in D$ *and* $(A_{x,y}, A_{y,z}) \in \mathcal{B}(x,y) \times \mathcal{B}(y,z)$

$$\frac{g(A_{y,z})}{g(A_{x,y})} \leq c \left(\frac{r(y,z)^\beta}{r(x,y)^\beta} \frac{\ell((r(x,y))^{-2})}{\ell((r(y,z))^{-2})} \vee 1\right).$$

Proof. Note that if $r(x,y) \geq \varepsilon_1$, $g(A_{y,z}) \leq C_1 = g(A_{x,y})$. We will consider three cases separately:
(a) $r(x,y) < \varepsilon_1$ and $r(y,z) \geq \varepsilon_1$: By Lemma 3.2.11, we have

$$\frac{g(A_{y,z})}{g(A_{r(x,y)}(Q_y))} \leq c\,C_1 \frac{\ell((r(x,y))^{-2})}{r(x,y)^\beta} \leq c\,C_1 \varepsilon_1^{-\beta} \left(\sup_{\varepsilon_1 \leq s \leq \mathrm{diam}(D)} \ell(s^{-2})\right) \frac{r(y,z)^\beta}{r(x,y)^\beta} \frac{\ell((r(x,y))^{-2})}{\ell((r(y,z))^{-2})}.$$

(b) $r(y,z) \leq r(x,y) < \varepsilon_1$: Then $A_{r(y,z)}(Q_y) \in D \cap B(Q_y, r(x,y))$. Thus by Lemma 3.2.5 we have $g(A_{r(y,z)}(Q_y)) \leq c g(A_{r(x,y)}(Q_y))$.
(c) $r(x,y) < r(y,z) < \varepsilon_1$: By Lemma 3.2.4,

$$\frac{g(A_{r(y,z)}(Q_y))}{g(A_{r(x,y)}(Q_y))} \leq c \frac{r(y,z)^\beta}{r(x,y)^\beta} \frac{\ell((r(x,y))^{-2})}{\ell((r(y,z))^{-2})}.$$

Now the conclusion of the lemma follows immediately from (3.2.5). □

Thus, by Lemmas 3.2.7 and 3.2.12, we get the following lemma.

Lemma 3.2.13. *There exists a constant $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ such that for every $x, y, z, w \in D$ and $(A_{x,y}, A_{z,w}, A_{x,w}) \in \mathcal{B}(x,y) \times \mathcal{B}(z,w) \times \mathcal{B}(x,w)$,*

$$\frac{g(y)g(z)g(A_{x,w})^2}{g(A_{x,y})^2 g(A_{z,w})^2} \le c \left(\frac{r(x,w)^\beta}{r(x,y)^\beta} \frac{\ell((r(x,y))^{-2})}{\ell((r(x,w))^{-2})} \vee 1 \right) \left(\frac{r(x,w)^\beta}{r(z,w)^\beta} \frac{\ell((r(z,w))^{-2})}{\ell((r(x,w))^{-2})} \vee 1 \right).$$

Lemma 3.2.14. *There exists a constant $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ such that for every $x, y, z, w \in D$ and $(A_{x,y}, A_{z,w}, A_{x,w}) \in \mathcal{B}(x,y) \times \mathcal{B}(z,w) \times \mathcal{B}(x,w)$,*

$$\frac{g(y)g(z)g(A_{x,w})^2}{g(A_{x,y})^2 g(A_{z,w})^2} \le c \left(\frac{r(y,z)^\beta}{r(x,y)^\beta} \frac{\ell((r(x,y))^{-2})}{\ell((r(y,z))^{-2})} \vee 1 \right) \left(\frac{r(y,z)^\beta}{r(z,w)^\beta} \frac{\ell((r(z,w))^{-2})}{\ell((r(y,z))^{-2})} \vee 1 \right).$$

Proof. From Lemma 3.2.9, we get

$$\begin{aligned} \frac{g(y)g(z)g(A_{x,w})^2}{g(A_{x,y})^2 g(A_{z,w})^2} &\le c_1 \frac{g(y)g(z)}{g(A_{x,y})^2 g(A_{z,w})^2} (g(A_{x,y})^2 + g(A_{y,z})^2 + g(A_{z,w})^2) \\ &= c_1 \left(\frac{g(y)g(z)}{g(A_{z,w})^2} + \frac{g(y)g(z)}{g(A_{x,y})^2} + \frac{g(y)g(z)g(A_{y,z})^2}{g(A_{x,y})^2 g(A_{z,w})^2} \right). \end{aligned}$$

By applying Lemma 3.2.7 to both y and z, we have that (3.2.7) is less than or equal to

$$\begin{aligned} &c_2 \frac{g(y)}{g(A_{z,w})} + c_2 \frac{g(z)}{g(A_{x,y})} + c_3 \left(\frac{g(A_{y,z})}{g(A_{x,y})} \right) \left(\frac{g(A_{y,z})}{g(A_{z,w})} \right) \\ &\le c_2 \frac{g(y)}{g(A_{z,w})} + c_2 \frac{g(z)}{g(A_{x,y})} + c_4 \left(\frac{r(y,z)^\beta}{r(x,y)^\beta} \frac{\ell((r(x,y))^{-2})}{\ell((r(y,z))^{-2})} \vee 1 \right) \left(\frac{r(y,z)^\beta}{r(z,w)^\beta} \frac{\ell((r(z,w))^{-2})}{\ell((r(y,z))^{-2})} \vee 1 \right), \end{aligned}$$

where we used Lemma 3.2.12. Moreover, by Lemmas 3.2.7 and 3.2.12,

$$\frac{g(y)}{g(A_{z,w})} = \left(\frac{g(y)}{g(A_{y,z})} \right) \left(\frac{g(A_{y,z})}{g(A_{z,w})} \right) \le c \left(\frac{r(y,z)^\beta}{r(z,w)^\beta} \frac{\ell((r(z,w))^{-2})}{\ell((r(y,z))^{-2})} \vee 1 \right)$$

and

$$\frac{g(z)}{g(A_{x,y})} = \left(\frac{g(z)}{g(A_{y,z})} \right) \left(\frac{g(A_{y,z})}{g(A_{x,y})} \right) \le c \left(\frac{r(y,z)^\beta}{r(x,y)^\beta} \frac{\ell((r(x,y))^{-2})}{\ell((r(y,z))^{-2})} \vee 1 \right).$$

Combining these, (3.2.7) and the inequality $(\frac{a}{b} \vee 1) + (\frac{a}{c} \vee 1) \le 2(\frac{a}{b} \vee 1)(\frac{a}{c} \vee 1)$, valid for all $a, b, c > 0$, we have finished the proof. □

Lemma 3.2.15. *Let $\psi(r) = \frac{r^\beta}{\ell(r^{-2})}$ and $M \in (0, \infty)$. Then there exists a constant $c = c(M, \ell, \beta) > 0$*

such that

$$\frac{\psi(a_2)}{\psi(b_2)} \le c\left(\frac{\psi(a_1)}{\psi(b_1)} \vee 1\right) \quad \textit{for every } 0 < a_1 \le a_2 \le 2a_1 \le M \textit{ and } 0 < b_1 \le b_2 \le M.$$

Proof. Since ℓ is slowly varying at ∞, by [8, Theorem 1.5.3] there exists $R_1 < M/2$ such that

$$\frac{s^\beta}{\ell(s^{-2})} \le 2\,\frac{r^\beta}{\ell(r^{-2})} \quad \text{and} \quad \frac{\ell(r^{-2})}{\ell((2r)^{-2})} \le 2 \quad \forall s < r \le R_1. \tag{3.2.7}$$

Note that $\psi : (0,\infty) \to (0,\infty)$ is locally bounded from above and below by positive constants.

If $a_1 \le R_1/2$, since $a_2 < 2a_1 \le R_1$, by (3.2.7), $\psi(a_2) \le 2^{\beta+2}\psi(a_1)$. If $a_1 > R_1/2$, by the local boundedness of ψ, $\psi(a_2) \asymp \psi(a_1)$.

Similarly, if $b_2 \le R_1$, since $b_1 \le b_2 \le R_1$, by (3.2.7), $2\psi(b_2) \ge \psi(b_1)$. If $b_2 > R_1$, by the local boundedness of ψ and (3.2.7), there exists a c_1 such that $\psi(b_2) \ge c_1\psi(b_1)$. The lemma clearly follows from these observations. □

Now we are ready to prove the main result of this section, which is a generalization of the main result in [35].

Theorem 3.2.16 (Generalized 3G theorem)**.** *Let* $\psi(r) := \frac{r^\beta}{\ell(r^{-2})}$. *Suppose that* D *is a bounded* κ*-fat open set with characteristics* (r_0, κ). *Then there exists a positive constant* $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi)$ *such that for every* $x, y, z, w \in D$

$$\frac{G_D(x,y)G_D(z,w)}{G_D(x,w)} \le c\left(\frac{\psi(|x-w|)\wedge\psi(|y-z|)}{\psi(|x-y|)} \vee 1\right)\left(\frac{\psi(|x-w|)\wedge\psi(|y-z|)}{\psi(|z-w|)} \vee 1\right)\frac{G(x,y)G(z,w)}{G(x,w)}. \tag{3.2.8}$$

Proof. Let

$$G(x,y,z,w) := \frac{G_D(x,y)G_D(z,w)}{G_D(x,w)} \quad \text{and} \quad H(x,y,z,w) := \frac{G(x,y)G(z,w)}{G(x,w)}.$$

If $|x-w| \le \delta_D(x) \wedge \delta_D(w)$, by Proposition 3.2.1(ii) and Theorem 3.1.1(i), $G_D(x,w) \ge cG(x,w)$. Thus by (3.2.1) and Theorem 3.1.1(i) we have $G(x,y,z,w) \le cH(x,y,z,w)$.

On the other hand, if $|y-z| \le \delta_D(y) \wedge \delta_D(z)$, then by Proposition 3.2.1(ii) and Theorem

3.1.1(i), $G_D(y,z) \geq cG(y,z)$. Using this and Theorem 3.2.10, we have that there exists a constant $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ such that

$$\begin{aligned} G(x,y,z,w) &= \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \frac{G_D(x,z)G_D(z,w)}{G_D(x,w)} \frac{1}{G_D(y,z)} \\ &\leq c \frac{G(x,y)G(y,z)}{G(x,z)} \frac{G(x,z)G(z,w)}{G(x,w)} \frac{1}{G(y,z)} = cH(x,y,z,w). \end{aligned}$$

Now we assume that $|x-w| > \delta_D(x) \wedge \delta_D(w)$ and $|y-z| > \delta_D(y) \wedge \delta_D(z)$. Since $\delta_D(x) \vee \delta_D(w) \leq \delta_D(x) \wedge \delta_D(w) + |x-w|$, using the assumption $\delta_D(x) \wedge \delta_D(w) < |x-w|$, we obtain $r(x,w) < 2|x-w|$. Similarly, $r(y,z) < 2|y-z|$. Let $A_{x,w} \in \mathcal{B}(x,w)$, $A_{x,y} \in \mathcal{B}(x,y)$ and $A_{z,w} \in \mathcal{B}(z,w)$. Applying Lemmas 3.2.13 and 3.2.14 to Theorem 3.2.3, we have

$$\begin{aligned} G(x,y,z,w) &\leq c \frac{g(y)g(z)g(A_{x,w})^2}{g(A_{x,y})^2 g(A_{z,w})^2} H(x,y,z,w) \\ &\leq c \left[\left(\frac{\psi(r(x,w))}{\psi(r(x,y))} \wedge \frac{\psi(r(y,z))}{\psi(r(x,y))} \right) \vee 1 \right] \left[\left(\frac{\psi(r(x,w))}{\psi(r(z,w))} \wedge \frac{\psi(r(y,z))}{\psi(r(z,w))} \right) \vee 1 \right] H(x,y,z,w). \end{aligned}$$

Now applying Lemma 3.2.15, we arrive at the conclusion of the theorem. □

3.3 Feynman-Kac Perturbations

Throughout this section D is a bounded κ-fat open set. In this section, we will first recall the Kato classes introduced in [15, 24, 25]. Then we apply the 3G theorem and generalized 3G theorem to establish some concrete sufficient conditions for these classes. Note that X^D is an irreducible transient symmetric Hunt process satisfying the assumption at the beginning of [15, Section 3.2].

Definition 3.3.1. *A function q is said to be in the class $\mathbf{S}_\infty(X^D)$ if for any $\varepsilon > 0$ there are a Borel subset $K = K(\varepsilon)$ of finite Lebesgue measure and a constant $\delta = \delta(\varepsilon) > 0$ such that*

$$\sup_{(x,z)\in(D\times D)\setminus\{x=z\}} \int_{D\setminus K} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |q(y)| dy \leq \varepsilon$$

and that, for all measurable set $B \subset K$ with $|B| < \delta$,

$$\sup_{(x,z)\in(D\times D)\setminus\{x=z\}} \int_B \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |q(y)|dy \le \varepsilon.$$

Definition 3.3.2. *Suppose F is a bounded function on $D \times D$ vanishing on the diagonal. Let*

$$q_{|F|}(x) := \int_D |F(x,y)|J(x,y)dy.$$

(1) F is said to be in the class $\mathbf{A}_\infty(X^D)$ if for any $\varepsilon > 0$ there are a Borel subset $K = K(\varepsilon)$ of finite Lebesgue measure and a constant $\delta = \delta(\varepsilon) > 0$ such that

$$\sup_{(x,w)\in(D\times D)\setminus\{x=w\}} \int_{(D\times D)\setminus(K\times K)} \frac{G_D(x,y)G_D(z,w)}{G_D(x,w)}|F(y,z)|J(y,z)dzdy \le \varepsilon$$

and that, for all measurable sets $B \subset K$ with $|B| < \delta$,

$$\sup_{(x,w)\in(D\times D)\setminus\{x=w\}} \int_{(B\times D)\cup(D\times B)} \frac{G_D(x,y)G_D(z,w)}{G_D(x,w)}|F(y,z)|J(y,z)dzdy \le \varepsilon.$$

(2) F is said to be in the class $\mathbf{A}_2(X^D)$ if $F \in \mathbf{A}_\infty(X^D)$ and if the function $q_{|F|}$ is in $\mathbf{S}_\infty(X^D)$.

Now we are going to use the 3G theorem and generalized 3G theorem to give some concrete sufficient conditions for $\mathbf{S}_\infty(X^D)$ and $\mathbf{A}_2(X^D)$. First we prove the following simple lemma.

Lemma 3.3.3. *There exists a positive constant $c = c(\alpha, d, \ell)$ such that*

$$\ell(|x-z|^{-2})|x-z|^{d-\alpha} \le c\left(\ell(|x-y|^{-2})|x-y|^{d-\alpha} + \ell(|y-z|^{-2})|y-z|^{d-\alpha}\right).$$

Proof. By symmetry, without loss of generality, we assume $|x-y| \le |y-z|$. Since ℓ is slowly varying at ∞, by [8, Theorem 1.5.3] there exists $R_1 > 0$ such that

$$s^{d-\alpha}\ell(s^{-2}) \le 2\, r^{d-\alpha}\ell(r^{-2}) \quad \text{and} \quad \ell((2r)^{-2}) \le 2\,\ell(r^{-2}) \quad \forall s < r \le R_1. \tag{3.3.1}$$

From (3.3.1), we see that

$$\ell(|x-z|^{-2})|x-z|^{d-\alpha} < c_1.$$

If $|y-z| \le R_1$, then $|x-z| \le |x-y| + |y-z| \le 2|y-z| \le 2R_1$. Thus by (3.3.1),

$$\ell(|x-z|^{-2})|x-z|^{d-\alpha} \le 2^{1+d-\alpha}\ell((2|y-z|)^{-2})|y-z|^{d-\alpha} \le 2^{2+d-\alpha}\ell(|y-z|^{-2})|y-z|^{d-\alpha}.$$

If $|y-z| > R_1$, by the local boundedness of ℓ and (3.3.1), we have

$$\ell(|x-z|^{-2})|x-z|^{d-\alpha} < c_1 < c_2\ell(|y-z|^{-2})|y-z|^{d-\alpha}.$$

□

Theorem 3.3.4. *A function q on D is in $\mathbf{S}_\infty(X^D)$ if*

$$\lim_{r\downarrow 0}\sup_{x\in D}\int_{|x-y|\le r}\frac{|q(y)|dy}{|x-y|^d\phi(|x-y|^{-2})} \;=\; 0. \tag{3.3.2}$$

Proof. Without loss of generality, we assume that q is a positive function on D. It follows from Theorem 3.2.10, (**H1**), Lemma 3.3.3, and the assumption on ℓ that for every $x, y, z \in D$ we have

$$\begin{aligned}\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} &\le c_1\frac{\phi(|x-z|^{-2})}{\phi(|x-y|^{-2})\phi(|y-z|^{-2})}\frac{|x-z|^d}{|x-y|^d|y-z|^d}\\ &\le c_2\left(\frac{1}{\phi(|x-y|^{-2})|x-y|^d}+\frac{1}{\phi(|y-z|^{-2})|y-z|^d}\right).\end{aligned} \tag{3.3.3}$$

We claim that a positive function q satisfying (3.3.2) is integrable on D. Let

$$M(r) := \sup_{w\in D}\int_{|w-y|\le r}\frac{q(y)dy}{|w-y|^d\phi(|w-y|^{-2})}.$$

By (**H1**) and [8, Theorem 1.5.3], there exists $s_0 > 0$ such that

$$u^d\phi(u^{-2}) \;\le\; 2s^d\phi(s^{-2}), \qquad u \le s \le s_0. \tag{3.3.4}$$

Then, using (3.3.2), we can choose $s_1 \le s_0$ such that $M(s_1) < \infty$. Now by (3.3.4),

$$\sup_{x\in D}\int_{|x-y|\le s_1} q(y)dy \le \sup_{x\in D}\int_{|x-y|\le s_1}\frac{2s_1^d\phi(s_1^{-2})q(y)dy}{|x-y|^d\phi(|x-y|^{-2})} \le 2s_1^d\phi(s_1^{-2})M(s_1) < \infty,$$

which implies that q is integrable on D.

By (3.3.3), we have for every Borel subset A of D and every $(x,z) \in D \times D$,

$$\int_A \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} q(y)dy \;\le\; 2\,c_2\,M(r) \;+\; 2\,c_2 \sup_{w\in D} \int_{A\cap B(w,r)^c} \frac{q(y)dy}{\phi(|w-y|^{-2})|w-y|^d}$$
$$\le 2\,c_2\,M(r) \;+\; \int_A q(y)dy \left(\sup_{s\in[r,\mathrm{diam(D)}]} \frac{2c_2}{\phi(s^{-2})s^d} \right) =: 2\,c_2\,M(r) + \left(\int_A q(y)dy\right) a(r).$$

Given ε, choose $r_1 = r_1(\varepsilon) \in (0, \mathrm{diam}(D))$ such that $2\,c_2\,M(r_1) < \varepsilon/2$ and let $\delta := 2^{-1}\varepsilon/a(r_1)$. This completes the proof of the theorem. □

The proof of the following theorem is similar to that of [35, Theorem 4.3].

Theorem 3.3.5. *If D is a bounded κ-fat open set with characteristics (r_0, κ) and F is a function on $D \times D$ with*

$$|F(x,y)| \;\le\; c_1 \frac{|x-y|^\epsilon}{\phi(|x-y|^{-2})} \tag{3.3.5}$$

for some $\epsilon > 0$ and $c_1 > 0$, then $F \in \mathbf{A}_2(X^D)$ and

$$\int_D \int_D \frac{G_D(x,y)G_D(z,w)}{G_D(x,w)} |F(y,z)| J(y,z) dy dz \le c_2 |x-w|^{\alpha+\epsilon} \phi(|x-w|^{-2}) \tag{3.3.6}$$

for some $c_2 > 0$.

Proof. We assume, without loss of generality, $\varepsilon < d-\alpha$. By the generalized 3G theorem (Theorem 3.2.16), there exists a positive constant $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi)$ such that

$$\frac{G_D(x,y)G_D(z,w)}{G_D(x,w)}$$
$$\le\; c_1 \left(\frac{\ell(|x-w|^{-2})}{\ell(|x-y|^{-2})\ell(|z-w|^{-2})} \frac{|x-w|^{d-\alpha}}{|x-y|^{d-\alpha}|z-w|^{d-\alpha}} + \frac{|x-w|^{d-\alpha+\beta}}{|x-y|^{d-\alpha+\beta}|z-w|^{d-\alpha}\ell(|z-w|^{-2})} \right.$$
$$\left. + \frac{|x-w|^{d-\alpha+\beta}}{|x-y|^{d-\alpha}|z-w|^{d-\alpha+\beta}\ell(|x-y|^{-2})} + \frac{|x-w|^{d-\alpha+2\beta}}{|x-y|^{d-\alpha+\beta}|z-w|^{d-\alpha+\beta}\ell(|x-w|^{-2})} \right).$$

Thus, by Theorem 3.1.1(ii) and (3.3.5), we have

$$\frac{G_D(x,y)G_D(z,w)}{G_D(x,w)} |F(y,z)| J(y,z) \le c_2 \sum_{i=1}^4 A_i(x,y,z,w)$$

where

$$\begin{aligned}
A_1(x,y,z,w) &:= \frac{\ell(|x-w|^{-2})}{\ell(|x-y|^{-2})\ell(|z-w|^{-2})}\frac{|x-w|^{d-\alpha}}{|x-y|^{d-\alpha}|z-w|^{d-\alpha}|y-z|^{d-\epsilon}},\\
A_2(x,y,z,w) &:= \frac{|x-w|^{d-\alpha+\beta}\ell(|z-w|^{-2})^{-1}}{|x-y|^{d-\alpha+\beta}|z-w|^{d-\alpha}|y-z|^{d-\epsilon}},\\
A_3(x,y,z,w) &:= \frac{|x-w|^{d-\alpha+\beta}\ell(|x-y|^{-2})^{-1}}{|z-w|^{d-\alpha+\beta}|x-y|^{d-\alpha}|y-z|^{d-\epsilon}},\\
A_4(x,y,z,w) &:= \frac{|x-w|^{d-\alpha+2\beta}\ell(|x-w|^{-2})^{-1}}{|x-y|^{d-\alpha+\beta}|z-w|^{d-\alpha+\beta}|y-z|^{d-\epsilon}}.
\end{aligned}$$

First let

$$c_3 := \sup_{(\widetilde{x},\widetilde{y})\in D\times D}\frac{|\widetilde{x}-\widetilde{y}|^{\alpha/2}}{\ell(|\widetilde{x}-\widetilde{y}|^{-2})} < \infty.$$

Then we have

$$\begin{aligned}
&\int_D\int_D A_1(x,y,z,w)dydz\\
=\ &\int_D\int_D \frac{\ell(|x-w|^{-2})}{\ell(|x-y|^{-2})\ell(|z-w|^{-2})}\frac{|x-w|^{d-\alpha}}{|x-y|^{d-\alpha}|z-w|^{d-\alpha}|y-z|^{d-\epsilon}}dydz\\
\le\ & c_3^2|x-w|^{d-\alpha}\ell(|x-w|^{-2})\int_D\int_D |x-y|^{-d+\frac{\alpha}{2}}|z-w|^{-d+\frac{\alpha}{2}}|y-z|^{-d+\epsilon}dydz\\
\le\ & c_3^2|x-w|^{\epsilon}\ell(|x-w|^{-2}) \le c_4|x-w|^{\alpha+\epsilon}\phi(|x-w|^{-2}).
\end{aligned}$$

The second to last inequality comes from [32, Lemma 3.12], and the last follows from (**H1**). Similar techniques can be applied to the case A_2, A_3, A_4 and this proves (3.3.6).

Now using Lemma 3.3.3, we have

$$\begin{aligned}
A_1(x,y,z,w) \le &\frac{1}{\ell(|z-w|^{-2})|z-w|^{d-\alpha}}\frac{1}{|y-z|^{d-\epsilon}} + \frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}}\frac{1}{|y-z|^{d-\epsilon}}\\
&+\frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}}\frac{1}{\ell(|z-w|^{-2})|z-w|^{d-\alpha}}\frac{1}{|y-z|^{\alpha-\epsilon}}.
\end{aligned}$$

Since $\varepsilon > 0$ and ℓ is slowly varying at ∞, the following two families

$$\{(y,z)\mapsto \ell(|x-y|^{-2})^{-1}|x-y|^{\alpha-d}|y-z|^{\epsilon-d},\ x\in D\},$$

$$\{(y,z)\mapsto \ell(|z-w|^{-2})^{-1}|z-w|^{\alpha-d}|y-z|^{\epsilon-d},\ w\in D\}$$

are uniformly integrable over cylindrical sets of the form $B\times D$ and $D\times B$, for any Borel set $B\subset D$. Now let us show that the following family of functions are uniformly integrable over cylindrical sets of the form $B\times D$ and $D\times B$:

$$\left\{(y,z)\mapsto \frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}\ell(|z-w|^{-2})|z-w|^{d-\alpha}|y-z|^{\alpha-\epsilon}},\ x,w\in D\right\}. \quad (3.3.7)$$

Let us consider the family (3.3.7) when the exponent of $|y-z|$ is negative, i.e., $\epsilon<\alpha$. Otherwise the family (3.3.7) is uniformly integrable since $|y-z|^{\epsilon-\alpha}<c$.

Applying Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}\ell(|z-w|^{-2})|y-z|^{\alpha-\epsilon}|z-w|^{d-\alpha}}\\
&=\left(\frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}\ell(|z-w|^{-2})|z-w|^{d-\alpha}}\right)\left(\frac{1}{|y-z|^{\alpha-\epsilon}}\right)\\
&\le\frac{1}{p}\left(\frac{1}{(\ell(|x-y|^{-2}))^{p}|x-y|^{(d-\alpha)p}(\ell(|z-w|^{-2}))^{p}|z-w|^{(d-\alpha)p}}\right)+\frac{1}{q}\left(\frac{1}{|y-z|^{(\alpha-\epsilon)q}}\right).
\end{aligned}$$

Since ℓ is slowly varying at ∞, it suffices to find $p,q>1$ satisfying $\frac{1}{p}+\frac{1}{q}=1$ and $(d-\alpha)p<d,(\alpha-\epsilon)q<d$. By choosing p in the interval

$$\left(\left(1\vee\frac{d}{d-\alpha+\epsilon}\right),\frac{d}{d-\alpha}\right),$$

we get that the family (3.3.7) is uniformly integrable. Note that this interval is not empty since $\dfrac{d}{d-\alpha+\epsilon}<\dfrac{d}{d-\alpha}$ by $(\alpha+\epsilon)\wedge d>\alpha$ and $\dfrac{d}{d-\alpha}>1$. Similar techniques can be applied to the case A_2, A_3, A_4 and this proves $F\in\mathcal{A}_\infty(X^D)$. (See [35, page 131–132].) Since

$$q_{|F|}(dx)=\int_D|F(x,y)|J(x,y)dy\le\int_D c\,|x-y|^{\epsilon-d}dy\le c,$$

it follows from Theorem 3.3.4 that $q_{|F|}\in\mathbf{S}_\infty(X^D)$ and therefore F is in $\mathbf{A}_2(X^D)$. □

For $w\in D$, we denote by $\mathbb{E}_x^w$ the expectation for the conditional process obtained from X^D through Doob's h-transform with $h(\cdot)=G_D(\cdot,w)$ starting from $x\in D$. For $q\in\mathbf{S}_\infty(X^D)$ and

$F \in \mathbf{A}_2(X^D)$, we define

$$e_{q+F}(t) := \exp\left(\int_0^t q(X_s^D)ds + \sum_{0<s\le t} F(X_{s-}^D, X_s^D)\right).$$

It gives rise to a Schrödinger semigroup

$$Q_t f(x) := \mathbb{E}_x\left[e_{q+F}(t) f(X_t^D)\right].$$

When $x \mapsto \mathbb{E}_x\left[e_{q+F}(\tau_D)\right]$ is bounded, it follows from [15, Theorem 3.9] that the Green function for the Schrödinger semigroup $\{Q_t,\, t \ge 0\}$ is

$$V_D(x,y) = \mathbb{E}_x^y\left[e_{q+F}(\tau_D)\right] G_D(x,y), \tag{3.3.8}$$

that is,

$$\int_D V_D(x,y) f(y)\, dy = \int_0^\infty Q_t f(x)\, dt = \mathbb{E}_x\left[\int_0^\infty e_{q+F}(t) f(X_t^D)\, dt\right]$$

for any Borel measurable function $f \ge 0$ on D.

Let $u(x,y) := \mathbb{E}_x^y\left[e_{q+F}(\tau_D)\right]$ for $y \in D$. Applying [15, Theorems 3.10] and [16, Theorems 3.4 and Section 6] (see also [24]) to our case, we get

Theorem 3.3.6. *Let $q \in \mathbf{S}_\infty(X^D)$ and $F \in \mathbf{A}_\infty(X^D)$ be such that the gauge function $x \mapsto \mathbb{E}_x\left[e_{q+F}(\tau_D)\right]$ is bounded. The following properties hold.*

(1) The conditional gauge function $u(x,y)$ is continuous on $(D \times D) \setminus \{(x,x) : x \in D\}$, hence by (3.3.8) so is $V_D(x,y)$.

(2) There exists a positive constant $c = c(\phi, D)$ such that

$$c^{-1}\, G_D(x,y) \;\le\; V_D(x,y) \;\le\; c\, G_D(x,y), \quad x,y \in D.$$

3.4 Green Function Estimate for Perturbations of Subordinate Brownian Motions

In this section, we consider Green function estimates for perturbations of subordinate Brownian motions. Throughout this section, Y is a symmetric Lévy process with a Lévy density $J^Y(x) := J(x) + \sigma(x)$ and we assume that there exist some constants $c > 0, \rho \in (0, d)$ such that

$$|\sigma(x)| \le c \max\{|x|^{-d+\rho}, 1\} \qquad \text{for } x \in \mathbb{R}^d. \tag{3.4.1}$$

Since $|\sigma(x)| \le J^Y(x) + J(x)$, clearly (3.4.1) implies that σ is integrable in $\mathbb{R}^d$. One particular example of Y is obtained with $J^Y(x) = J(x) 1_{B(0,1)}(x)$.

First we show that the transition density of Y is in $C_b^\infty(\mathbb{R}^d)$, where $C_b^\infty(\mathbb{R}^d)$ is the set of smooth and bounded functions on $\mathbb{R}^d$.

Lemma 3.4.1. *The process Y has a transition density $p^Y(t,x,y) = p^Y(t, y-x)$ such that $x \to p^Y(t,x)$ is in $C_b^\infty(\mathbb{R}^d)$ for each $t > 0$.*

Proof. The Lévy exponent of Y is given by

$$\Psi^Y(\xi) = \Psi(\xi) + \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\xi, x))\sigma(x)dx.$$

Since

$$\Big| \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\xi, x))\sigma(x)dx \Big| \le 2|\sigma|_{L^1(\mathbb{R}^d)}, \tag{3.4.2}$$

we have $\int |\exp(-t\Psi^Y(\xi))||\xi|^n d\xi < \infty$ for every $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. Note that for $t > 0$

$$p^Y(t,x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} e^{-t\Psi^Y(\xi)} d\xi \le (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\Psi^Y(\xi)} d\xi = p^Y(t,0) < \infty.$$

Now the assertion of the lemma follows immediately. □

For any open set U, we will use τ_U^Y to denote the first time Y exits U, i.e., $\tau_U^Y = \inf\{t > 0 : Y_t \notin U\}$. The killed process of Y in U is denoted by Y^U. It follows easily from [50, Lemma 48.3] that for any bounded open subset U, there exists $t_1 > 0$ such that $\sup_{x \in \mathbb{R}^d} \mathbb{P}_x(Y_{t_1} \in U) < 1$. Put

$\theta := \sup_{x\in\mathbb{R}^d} \mathbb{P}_x(\tau_U^Y > t_1) \le \sup_{x\in\mathbb{R}^d} \mathbb{P}_x(Y_{t_1} \in U) < 1$. Then by the Markov property and an induction argument,

$$\sup_{x\in\mathbb{R}^d} \mathbb{P}_x(\tau_U^Y > nt_1) \le \theta^n.$$

Thus

$$\sup_{x\in U} \mathbb{E}_x[\tau_U^Y] \le \frac{t_1}{1-\theta} < \infty. \tag{3.4.3}$$

Now we state some auxiliary properties of $p^X(t,x)$. We need these properties only when we prove the (killed) heat kernel $p_D^Y(t,x)$ is continuous and it will not be needed in the rest of the thesis.

Lemma 3.4.2. *There exist constants $c > 0$ and $\zeta > 0$ such that $p^X(t,x) \le ct^{-\zeta}$ for every $t \in (0,1]$.*

Proof. The heat kernel $p^X(t,x)$ can be expressed in terms of Fourier transforms by $p^X(t,x) = (2\pi)^{-d}\int e^{-i\xi\cdot x}e^{-t\Psi(\xi)}d\xi$. Since ℓ is slowly varying at ∞ there is a constant c_1 such that $|\xi|^\alpha \ell(|\xi|^2) \ge c_1|\xi|^{\alpha/2}$ for $|\xi| \ge 1$. From this it follows that for $t \in (0,1]$

$$\begin{aligned} p^X(t,x) &\le p^X(t,0) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-t\Psi(\xi)}d\xi \le (2\pi)^{-d}\int_{|\xi|<1} 1d\xi + (2\pi)^{-d}\int_{|\xi|\ge 1} e^{-tc_1|\xi|^{\frac{\alpha}{2}}}d\xi \\ &\le (2\pi)^{-d}\frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} + c_2t^{-\frac{2d}{\alpha}} \le c_3t^{-\frac{2d}{\alpha}}. \end{aligned}$$

□

Lemma 3.4.3. *For every $\delta > 0$ there exists a constant $c = c(\delta)$ such that for every $|x| \ge \delta$ and $t > 0$*

$$p^X(t,x) \le c(\delta), \tag{3.4.4}$$

$$|\sigma(x) + \left(p^X(t,\cdot) * \sigma\right)(x)| \le c(\delta).$$

Proof. The heat kernel $p^X(t,x)$ can also be written as $p^X(t,x) = \int_0^\infty (4\pi s)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4s}} \mathbb{P}(S_t \in ds)$ and thus $p^X(t,x) < c_1(\delta)$ for all $|x| \ge \delta$ and $t > 0$. Next since σ is integrable on $\mathbb{R}^d$ and uniformly bounded away from 0, it follows from (3.4.4) that for $|x| \geq \delta$ and $t > 0$

$$\begin{aligned} & p^X(t,\cdot) * \sigma(x) = \int p^X(t, x-y)\sigma(y)dy \\ = & \int_{|x-y|\geq \delta/2} p^X(t,x-y)\sigma(y)dy + \int_{|x-y|<\delta/2} p^X(t,x-y)\sigma(y)dy \\ \leq & c_1(\delta)\|\sigma\|_{L^1(\mathbb{R}^d)} + \|\sigma\|_{L^\infty(B(0,\delta/2)^c)} \int_{|x-y|<\delta/2} p^X(t,x-y)dy \leq c_2(\delta) < \infty. \end{aligned}$$

□

In the remainder of this section ζ will stand for the constant in Lemma 3.4.2. Using Lemmas 3.4.2 and 3.4.3, the proof of the next lemma is the same as that of [32, Lemma 2.6], so we omit the proof.

Lemma 3.4.4. *For every δ there exists a constant $c = c(\delta, \zeta) > 0$ such that $p^Y(t,x) \leq c$ for $|x| \geq (1 \vee [\zeta])\delta$ and $t > 0$.*

Now we prove that $p_D^Y(t,\cdot,\cdot)$ is jointly continuous for any bounded open set D.

Lemma 3.4.5. *For any bounded open set D, $p_D^Y(t,\cdot,\cdot)$ is jointly continuous on $D \times D$.*

Proof. By Lemmas 3.4.2, 3.4.3, and 3.4.4, we have for every $T, L > 0$

$$\sup_{|x-y|\geq L, 0<t\leq T} p^Y(t,x,y) < \infty. \tag{3.4.5}$$

By the strong Markov property and the continuity of $p^Y(t,\cdot,\cdot)$, the transition density $p_D^Y(t,x,y)$ of Y^D for any open set D can be written as

$$p_D^Y(t,x,y) := p^Y(t,x,y) - \mathbb{E}_x[\, p^Y(t-\tau_D^Y, Y_{\tau_D^Y}, y) : \tau_D^Y \leq t] \quad \text{for } t > 0, x, y \in \mathbb{R}^d. \tag{3.4.6}$$

Now using (3.4.5) and (3.4.6) and following the routine argument (see [27]), one can show that for any open set D, $p_D^Y(t,\cdot,\cdot)$ is jointly continuous in $D \times D$. □

In the remainder of this section we will show that, for any bounded κ-fat open domain D, G_D^Y is comparable to G_D, the Green function of X^D. We will accomplish this by first dealing with the case σ is positive, then the general case.

3.4.1 Positive σ Case

Assume Z is a symmetric Lévy process with a Lévy density $J^Z(x) := J(x) + \widetilde{\sigma}(x)$ and we assume that there exist some constants $c > 0, \rho \in (0, d)$ such that

$$0 \le \widetilde{\sigma}(x) \le c \max\{|x|^{-d+\rho}, 1\} \qquad \text{for } x \in \mathbb{R}^d. \tag{3.4.7}$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ of X is given by

$$\begin{aligned} \mathcal{E}(u,v) &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x)-u(y))(v(x)-v(y))J(x,y)dxdy, \\ \mathcal{F} &= \{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u,u) < \infty\}. \end{aligned}$$

Another expression for $\mathcal{E}$ is given by

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^d} \hat{u}(\xi)\bar{\hat{v}}(\xi)\Psi(\xi)d\xi,$$

where $\hat{u}$ is the Fourier transform of u. The Dirichlet form $(\mathcal{E}^Z, \mathcal{F}^Z)$ of Z is given by

$$\begin{aligned} \mathcal{E}^Z(u,v) &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x)-u(y))(v(x)-v(y))J^Z(x,y)dxdy, \\ \mathcal{F}^Z &= \{u \in L^2(\mathbb{R}^d) : \mathcal{E}^Z(u,u) < \infty\}. \end{aligned}$$

Another expression for $\mathcal{E}^Z$ is given by

$$\mathcal{E}^Z(u,v) = \int_{\mathbb{R}^d} \hat{u}(\xi)\bar{\hat{v}}(\xi)\Psi^Z(\xi)d\xi$$

where $\Psi^Z(\xi) = \Psi(\xi) + \int_{\mathbb{R}^d\setminus\{0\}}(1 - \cos(\xi, x))\widetilde{\sigma}(dx)$. It follows from (3.4.2) that there exists $c > 0$ such that

$$c^{-1}\mathcal{E}_1(u,u) \le \mathcal{E}_1^Z(u,u) \le c\mathcal{E}_1(u,u).$$

Therefore we know that $\mathcal{F}^Z = \mathcal{F}$ and that a set is of zero capacity for X if and only if it is of zero capacity for Z.

In the remainder of this subsection, we always assume that D is a bounded κ-fat set. The

Dirichlet forms of X^D and Z^D are given by $(\mathcal{E}, \mathcal{F}_D)$ and $(\mathcal{E}^Z, \mathcal{F}_D^Z)$ respectively, where

$$\mathcal{F}_D = \mathcal{F}_D^Z = \{u \in \mathcal{F} | u = 0 \text{ on } D^c \text{ except for a set of zero capacity}\}.$$

For $u, v \in \mathcal{F}_D$, we have

$$\begin{aligned} \mathcal{E}(u,v) &= \frac{1}{2}\int_D \int_D (u(x)-u(y))(v(x)-v(y))J(y-x)dxdy + \int_D u(x)v(x)\kappa_D(x)dx, \\ \mathcal{E}^Z(u,v) &= \frac{1}{2}\int_D \int_D (u(x)-u(y))(v(x)-v(y))J^Z(y-x)dxdy + \int_D u(x)v(x)\kappa_D^Z(x)dx, \end{aligned}$$

where $\kappa_D(x) = \int_{D^c} J(y-x)dy$ and $\kappa_D^Z(x) = \int_{D^c} J^Z(y-x)dy = \kappa_D^X(x) + \int_{D^c} \widetilde{\sigma}(y-x)dy$. Define $F(x,y) := \frac{J^Z(y-x)}{J(y-x)} - 1 = \frac{\widetilde{\sigma}(y-x)}{J(y-x)}$ and $q(x) := \kappa_D(x) - \kappa_D^Z(x)$. Note that $\inf_{x,y \in D} F(x,y) \geq 0$. Now define

$$K_t = \exp(\sum_{0<s\leq t} \ln(1 + F(X_{s-}^D, X_s^D)) - \int_0^t \int_D F(X_s^D, y)J(y - X_s^D)dyds + \int_0^t q(X_s^D)ds)$$

and

$$Q_t f(x) := \mathbb{E}_x[K_t f(X_t^D)], \quad x \in D.$$

By calculating the quadratic form of Q_t using techniques similar to those on [26, p. 275], one can see that Q_t is the semigroup associated with the Dirichlet form $(\mathcal{E}^Z, \mathcal{F}_D^Z)$.

By using Theorem 3.1.1 and the assumption on $\widetilde{\sigma}$, it is easy to see there exist $\epsilon > 0$ and $c' > 0$ such that $F(x,y) \leq c' \frac{|x-y|^\epsilon}{\phi(|x-y|^{-2})}$ for all $x, y \in D$. (For example, we can take $\epsilon = \frac{\rho}{2}$.) Thus, by Theorem 3.3.5, the function $F(x,y) \in \mathbf{A}_2(X^D)$. Since $|q(x)| = |-\int_{D^c} \sigma(y-x)dy| \leq \int_{\mathbb{R}^d} \sigma(z)dz < \infty$, we know that $q \in S_\infty(X^D)$ by Theorem 3.3.4.

Note that the killing intensity κ_D^Z of Z^D is bounded from below by a positive constant so it follows that

$$\inf\{\mathcal{E}^Z(u,u) : u \in \mathcal{F}_D^Z \text{ with } \int_D u(x)^2 dx = 1\} > 0.$$

This implies that $\int_0^\infty Q_t dt$ is a bounded operator in $L^2(D, dx)$ and so for any Borel subset $B \subset D$,

$$\int_0^\infty Q_t 1_B(x)dt = \mathbb{E}_x[\int_0^\infty K_t 1_B(X_t^D)dt] < \infty, \qquad \text{for all } x \in D. \tag{3.4.8}$$

It follows from (3.4.3) and [31, Proposition 2.2 (ii)] that the Green function $G^Z_D(\cdot,\cdot)$ of Z^D exists and strictly positive on $D\times D$ for any bounded open set D. Moreover, since Z satisfies the condition $(A1)$ in [39], it follows from [31, Proposition 2.1], [39, Theorem 3.11] and our Lemmas 3.4.2 and 3.4.5 that the semigroup of Z^D is intrinsically ultracontractive, that is there exits a constant $c_1 = c_1(D,t)$ such that $p^Z_D(t,x,y) \le c_1\phi_1(x)\phi_1(y)$, where ϕ_1 is the eigenfunction of semigroup of Z^D associated with the largest eigenvalue $\lambda_1 < 0$ of the generator of Z^D and $\|\phi_1\|_{L^2(D)} = 1$. Furthermore it follows from [39, Theorem 3.13] there is a constant $c_2 > 0$ such that $p^Z_D(t,x,y) \le c_2 e^{\lambda_1 t}\phi_1(x)\phi_1(y)$ for all $t > 1$. Hence by Lemma 3.4.4, the dominated convergence theorem and the continuity of $p^Z_D(t,\cdot,\cdot)$, $G^Z_D(\cdot,\cdot)$ is continuous on $(D\times D)\setminus\{x=y\}$. Now, from (3.4.8), Theorems 3.3.4 and 3.3.5, we know that the assumptions of Theorem 3.3.6 are satisfied. Since the Green function of the semigroup Q_t is $G^Z_D(x,y) = G_D(x,y)\mathbb{E}^y_x[K_{\tau_D}]$, the following result is an immediate consequence of Theorem 3.3.6.

Theorem 3.4.6. *If Z is a purely discontinuous symmetric Lévy process with Lévy density $J^Z(x) = J(x) + \widetilde{\sigma}(x)$ satisfying (3.4.7) and D be a bounded κ-fat open set in $\mathbb{R}^d$. Then the Green function $G^Z_D(x,y)$ for Z in D is continuous on $(D\times D)\setminus\{(x,x): x\in D\}$. Moreover, there is a constant $c = c(D,d,\phi) > 0$ such that*

$$c^{-1}\, G_D(x,y) \;\le\; G^Z_D(x,y) \;\le\; c\, G_D(x,y), \quad x,y\in D.$$

3.4.2 General Case

Now we return to the general case where σ can take both signs. From now on we assume D is a bounded κ-fat domain (connected open set). Let Z be the Lévy process with a Lévy density $J^Z(x) := J^Y(x)\vee J(x)$. Then $\widetilde{\sigma}(x) := J^Z(x) - J(x)$ satisfies (3.4.7). By Lemma 3.4.5, $p^Y_D(t,\cdot,\cdot)$ and $p^Z_D(t,\cdot,\cdot)$ are jointly continuous on $D\times D$. Note that [39, Condition (A1)(b)] is true for all three processes X, Y and Z. Since D is a domain, by following the argument in the proof of [31, Proposition 2.2], one can show that $p^X_D(t,\cdot,\cdot)$, $p^Y_D(t,\cdot,\cdot)$ and $p^Z_D(t,\cdot,\cdot)$ are strictly positive for all $t > 0$. Thus [32, Property A] is valid. (Also see [39, Corollary 3.12].) Using an argument similar to the one in the paragraph before Theorem 3.4.6, we see that $G^Y_D(\cdot,\cdot)$ and $G^Z_D(\cdot,\cdot)$ are strictly positive and jointly continuous on $D\times D$. Now it follows from [32, Theorem 3.1] and the joint continuity of $G^Y_D(x,y)$ that for every bounded κ-fat domain D

$$G^Y_D(x,y) \le c_1 G^Z_D(x,y) \le c_2 G_D(x,y), \tag{3.4.9}$$

for some constants $c_1 = c_1(d,D,\phi)$ and $c_2 = c_2(d,D,\phi)$.

In the remainder of this subsection we will show that $G^Y_D(x,y) \ge c_3 G_D(x,y)$ for some $c_3 > 0$. We will follow the argument in [32] closely.

By [32, Lemma 2.4], for any bounded open set D, $\mathbb{E}_x[\tau^Z_D] \asymp \mathbb{E}_x[\tau^Y_D]$ and $\mathbb{E}_x[\tau^Z_D] \asymp \mathbb{E}_x[\tau_D]$. Thus

Lemma 3.4.7. *For any bounded open set D, we have $\mathbb{E}_x[\tau_D] \asymp \mathbb{E}_x[\tau^Y_D]$.*

The following result is similar to [34, Lemma 17]. Recall that the function g is defined in Section 3.2.

Lemma 3.4.8. *Let D be a bounded κ-fat domain. Then*

$$g(x) \asymp \mathbb{E}_x[\tau_D].$$

Proof. Pick a point $z \in D^c$ such that $\delta_D(z) = \text{diam}(D) + 1$ and let $B := B(z,1)$. Consider the function $f(x) := \mathbb{P}_x(X_{\tau_D} \in B)$. By the Lévy system of X, we know that $f(x) = \int_B \int_D G_D(x,y) J(z-y) dy dz$. For $y \in D, z \in B$, $\text{diam}(D) < |y-z| < 2\text{diam}(D)+2$, so by monotonicity of j, $j(2\text{diam}(D)+2)|B| \cdot \mathbb{E}_x[\tau_D] \le f(x) \le j(\text{diam}(D))|B| \cdot \mathbb{E}_x[\tau_D]$. Since $g(x)$ is equal to $G_D(x,z_0)$ on $|x-z_0| > \frac{\delta_D(z_0)}{2}$, the assertion of this lemma now follows from Theorem 3.1.5. □

Lemma 3.4.9. *Let D be a bounded κ-fat domain and $\theta > 0$ a constant. If $x,y \in D$ satisfy $|x-y| \ge \theta$, then there is a constant $c = c(\theta,\phi,D,d)$ such that $G_D(x,y) \le c\,\mathbb{E}_x[\tau_D]\mathbb{E}_y[\tau_D]$.*

Proof. The proof of this lemma is similar to that of [32, Corollary 3.11]. By Theorem 3.2.3, we have

$$G_D(x,y) \le c_1 \frac{g(x)g(y)}{g(A)^2 |x-y|^d \phi(|x-y|^{-2})},$$

where $A \in \mathcal{B}(x,y)$. Since $\delta_D(A) \ge \frac{\kappa}{2} r(x,y) \ge \frac{\kappa}{2}|x-y| \ge \frac{\kappa\theta}{2}$, it follows from [41, Lemma 4.2] that

$$g(A) \asymp \mathbb{E}_A[\tau_D] \ge \mathbb{E}_A[\tau_{B(A,\frac{\kappa\theta}{2})}] \ge c_2 \frac{1}{\phi((\frac{\kappa\theta}{4})^{-2})}.$$

Now the theorem follows from Lemma 3.4.8. □

Recall that Y also satisfies [32, Property A] for the bounded κ-fat domain D, i.e.,

$$c\,\mathbb{E}_x[\tau_D^Y]\,\mathbb{E}_y[\tau_D^Y] \;\leq\; G_D^Y(x,y). \tag{3.4.10}$$

The following result says that the Green functions $G_D(x,y)$ and $G_D^Y(x,y)$ are comparable when the distance between x and y is not too small.

Theorem 3.4.10. *Let D be a bounded κ-fat domain and $\theta > 0$ a constant. If $x, y \in D$ satisfy $|x-y| \geq \theta$, there is a constant $c = c(\theta, \phi, D, d)$ such that $G_D(x,y) \;\leq\; c\,G_D^Y(x,y)$.*

Proof. It follows from (3.4.10), Lemmas 3.4.9 and 3.4.7 that

$$G_D(x,y) \leq c_1\mathbb{E}_x[\tau_D]\mathbb{E}_y[\tau_D] \leq c_2\mathbb{E}_x[\tau_D^Y]\mathbb{E}_y[\tau_D^Y] \leq c_3 G_D^Y(x,y).$$

□

Now we are going to prove that $G_D(x,y) \;\leq\; c\,G_D^Y(x,y)$ for some $c = c(d, D, \phi) > 0$ when x and y are close to each other. The next lemma is adapted from [32, Lemma 3,5 and Corollary 3.6] which use the proofs of [49, Lemmas 7 and 9]. In fact, the proofs of [49, Lemmas 7 and 9] work for a large class of Lévy processes including our Y and Z. Thus, we omit the proof.

Lemma 3.4.11. *For any bounded open set D, we have for any $x, w \in D$,*

$$G_D^Z(x,w) \leq G_D^Y(x,w) + \int_D \int_D G_D^Y(x,y)\sigma(y-z)G_D^Z(z,w)dydz.$$

Theorem 3.4.12. *For every bounded κ-fat domain D, there are constants $\delta = \delta(d, \phi, D, \sigma, \rho) > 0$ and $c = c(d, \phi, D, \sigma, \rho) > 0$ such that for all $x, w \in D$ with $|x-w| < \delta$, we have*

$$G_D(x,w) \;\leq\; c\,G_D^Y(x,w).$$

Proof. By Theorem 3.4.6, Lemma 3.4.11, and (3.4.9) there exist constants $c_i = c_i(d, \phi, D, \sigma, \rho)$,

$i = 1, 2$, such that

$$\begin{aligned} G_D(x,w) &\le c_1 G_D^Z(x,w) \le c_1 G_D^Y(x,w) + c_1 \int_D \int_D G_D^Y(x,y)\sigma(y-z) G_D^Z(z,w) dydz \\ &\le c_1 G_D^Y(x,w) + c_2 \int_D \int_D G_D(x,y)\sigma(y-z) G_D(z,w) dydz \\ &= c_1 G_D^Y(x,w) + c_2 G_D(x,w) \int_D \int_D \frac{G_D(x,y)G_D(z,w)}{G_D(x,w)} \frac{\sigma(y-z)}{J(y-z)} J(y-z) dydz. \end{aligned}$$

Since $\frac{\sigma(y-z)}{J(y-z)} \le c_3 \frac{|y-z|^\rho}{\phi(|y-z|^{-2})}$, by Theorem 3.3.5, there exists a $c_4 > 0$ such that

$$G_D(x,w) \le c_1 G_D^Y(x,w) + c_4 |x-w|^{\alpha+\rho} \phi(|x-w|^{-2}) G_D(x,w).$$

Now take δ small so that $c_4 |x-w|^{\alpha+\rho}\phi(|x-w|^{-2}) G_D(x,w) \le \frac{1}{2} G_D(x,w)$ if $|x-w| < \delta$. □

Combining (3.4.9), Theorems 3.4.10 and 3.4.12, we have proved the next theorem which is the main result of this chapter.

Theorem 3.4.13. *Suppose that $\alpha \in (0, 2 \wedge d)$ and D is a bounded κ-fat open domain. If Y is a symmetric Lévy process with a Lévy density $J^Y(x) := J(x) + \sigma(x)$ with σ satisfying the condition* (3.4.1)*, then the Green function G_D^Y of Y^D is comparable to the Green function G_D^X of X^D, i.e., there exists a constant $c = c(D, d, \phi, \rho, \sigma)$ such that*

$$c^{-1} G_D^Y(x,y) \le G_D(x,y) \le c G_D^Y(x,y), \quad x, y \in D.$$

Remark 3.4.14. *The condition that D is connected is crucial in Theorem 3.4.13. For example, if Y has a Lévy density $\nu^Y(x) = \nu(x) 1_{\{|x|<1\}}$ and $D = B(z,1) \cup B(w,1)$ where $z, w \in \mathbb{R}^d$, $|z-w| > 2$, then $G_D(x,y) > 0$ for $x, y \in D$ whereas $G_D^Y(x,y) = 0$ for $x \in B(z,1)$ and $y \in B(w,1)$.*

Combining the above theorem with the main result in [43] ([43, Theorem 1.1]), we immediately get the following.

Corollary 3.4.15. *Suppose that the assumptions of Theorem 3.4.13 are valid and further that D is a bounded $C^{1,1}$ domain, then the Green function $G_D^Y(x,y)$ satisfies*

$$G_D^Y(x,y) \asymp \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}} \right) \frac{1}{|x-y|^d \, \phi(|x-y|^{-2})}.$$

Chapter 4

Boundary Harnack Principle

In this chapter, we prove the boundary Harnack principle for nonnegative harmonic functions with respect to perturbations of subordinate Brownian motions that vanish outside a small ball and a part of the boundary of the domain. The boundary Harnack principle (BHP) for jump processes (or equivalently for non-local operators) is first established in [14] for symmetric stable processes in bounded Lipschitz domains and since then a lot of generalizations have been established. In one direction, the BHP is proved for more general domains than Lipschitz domains. It is established in [54] for bounded κ-fat open sets which are discontinuous analogues of John domains and in [12], the BHP is proved for arbitrary open sets with respect to rotationally symmetric stable processes and this version of the BHP is known as the uniform boundary Harnack principle. In the other direction, the BHP is proved for harmonic functions with respect to wider classes of processes than symmetric stable processes. In [40, 41], the BHP is established for a wide class of subordinate Brownian motions that include many interesting examples such as stable processes, an independent sum of stable processes, and relativistic stable processes. In [42], the uniform BHP is proved for a large class of Lévy processes which include subordinate Brownian motions considered in [40, 41] for arbitrary open sets. Also in [37, 38] the authors proved several versions of the BHP for so called truncated stable processes and the starting point of our research in this chapter was to generalize the result considered in [37, 38] to more general processes than truncated stable processes.

Now we will precisely state the main result of this chapter. In this chapter we will prove that for nonnegative harmonic functions with respect to perturbations of subordinate Brownian motions considered in the previous chapter that vanish outside a small ball and near a part of the boundary of the domain, the ratio of such harmonic functions remains bounded near the boundary of the domain. Let us state this theorem more precisely. Recall that the processes Y are rotationally symmetric Lévy processes with the Lévy density $J^Y(x) = j^Y(|x|)$ satisfying

(1) $\sigma(x) := J^Y(x) - J^X(x)$ is integrable in $\mathbb{R}^d$ and $\sigma(x) \le c|x|^{-d+\rho}$ for some constants $0 < \rho < d$, $c > 0$, and $x \in B(0,1)$.

(2) $\sigma(x)$ is bounded outside the unit ball $B(0,1)$.

From the condition (1) and Theorem 3.1.1, it is easy to see that there exists a positive constant c such that

$$J^Y(x) \le c_1 J^X(x), \quad x \in B(0,1).$$

From the condition (1) there is a constant c_2 such that $J^Y(x) \ge c_2 J^X(x), x \in B(0,r)$ for some $r > 0$. The exact value of r is not important and we will assume that $r = 1$. Hence we conclude that there is a constant $c > 0$ such that

$$c^{-1} J^X(x) \le J^Y(x) \le c J^X(x), \quad x \in B(0,1). \tag{4.0.1}$$

A typical example of Y satisfiying conditions mentioned above is the Lévy process whose Lévy density is equal to that of X on the closed unit ball $\overline{B(0,1)}$ and equal to 0 outside the closed unit ball. Note that in this case, Y are the so called the *truncated* SBMs, which is a natural generalization of truncated stable processes considered in [37, 38]. Now we state the main theorem of the chapter.

Theorem 4.0.16. *Suppose that D is a bounded κ-fat domain with characteristics (R,κ). Then there exists a constant R_1 such that if $r \le R_1$ and $Q \in \partial D$ such that for any nonnegative functions u, v which are regular harmonic in $D \cap B(Q,2r)$ with respect to Y and vanish in $(D^c \cap B(Q,2r)) \cup B(Q,1-2r)^c$, we have*

$$c^{-1}\frac{u(A_r(Q))}{v(A_r(Q))} \le \frac{u(x)}{v(x)} \le c\frac{u(A_r(Q))}{v(A_r(Q))}, \quad x \in D \cap B(Q, \frac{r}{2})$$

for some constant $c = c(D,d,\alpha,\ell,\sigma) > 1$.

The organization of this chapter is as follows. In section 4.1, we generalize the main result of chapter 3. We prove that for bounded κ-fat domains D, the Green functions $G_D^X(x,y)$ and $G_D^Y(x,y)$ are comparable uniformly for all D as long as the Lebesgue measure $|D|$ is small enough. Note that in [37, 38] the authors proved a similar result about the comparability of two Green functions of rotationally symmetric stable processes and truncated symmetric stable processes but the argument of [37, 38] depends on scaling invariant property of rotationally symmetric stable processes. In the case of subordinate Brownian motions, the corresponding scaling invariant property is not true anymore. We overcome this by showing that the constant C that satisfies the relation $C^{-1}G_D^X(x,y) \le G_D^Y(x,y) \le CG_D^X(x,y)$, $x, y \in D$ depends on D only via $|D|$, the Lebesgue measure of D and if $\widetilde{D}$ is another κ-fat domain with $|\widetilde{D}| \le |D|$ then the same constant $C = C(D)$ works for $\widetilde{D}$ and the relation $C^{-1}G_{\widetilde{D}}^X(x,y) \le G_{\widetilde{D}}^Y(x,y) \le CG_{\widetilde{D}}^X(x,y)$, $x, y \in \widetilde{D}$ is true. In section 4.2, we establish a version of the Harnack principle for nonnegative harmonic functions with respect to processes Y that vanish outside a small ball. In section 4.3, we establish a version of the BHP for nonnegative harmonic functions with respect to processes Y that vanish outside a small ball and a part of the boundary of the domain. Note that this version of the BHP is slightly weaker than the ordinary version of the BHP in a sense that we require that the harmonic functions vanish outside a small ball as well as a part of the boundary of the domain. This condition is not just technical. In fact in [37] the authors proved that the BHP fails for nonnegative harmonic functions with respect to truncated symmetric stable processes in non-convex domains. This indicates that the potential theory of perturbations of subordinate Brownian motions could be more delicate than those of subordinate Brownian motions.

4.1 Uniform Green Functions Comparability

Recall that X, Y are subordinate Brownian motions and their perturbations defined in the previous chapter. In this section, we prove that the Green functions $G_D^X(x,y)$, $G_D^Y(x,y)$ are comparable with an absolute constant C for all sufficiently small κ-fat open domains D. More precisely we will prove

Theorem 4.1.1. *Let $R > 0$. There exist constants $C = C(d, \kappa, X, Y, R)$ such that for any bounded κ-fat domain D with $|D| \le R$, we have*

$$C^{-1}G_D^Y(x,y) \le G_D^X(x,y) \le CG_D^Y(x,y), \quad x, y \in D. \tag{4.1.1}$$

We emphasize that the constant C works for all κ-fat domain D as long as $|D|$ is small enough. In general if D is a κ-fat open set with characteristics (κ, r_0), rD is a κ-fat domain with characteristics (κ, rr_0). In order to prove a scaling invariant version of the boundary Harnack principle for Y, we need a constant in (4.1.1) to be scaling invariant. In [37, 38] the authors considered truncated stable processes and achieved (4.1.1) when X are symmetric stable processes and Y are truncated stable processes by using the 3G theorem and the scaling invariant property of symmetric stable processes. Note that in the previous chapter, (4.1.1) is proved when X is subordinate Brownian motion but the constant depends on D via its characteristic (κ, r_0) and it is not clear there if the constant can be chosen uniformly for all sufficiently small κ-fat domains D.

We starts with some simple lemmas.

Lemma 4.1.2. *Let $\Phi^X(\xi)$ be the characteristic exponent of subordinate Brownian motions X. Then $\Phi^X(\xi) = 0$ if and only if $\xi = 0$.*

Proof. This is a simple consequence of the Bernstein function and the proof follows easily from (2.2.1) and the fact that $\Phi^X(\xi) = \phi(|\xi|^2)$. □

The next lemma is about the long time behavior of the heat kernel of X.

Lemma 4.1.3. *Let $p^X(t, x)$ be a heat kernel of X. Then*

$$\lim_{t\to\infty} p^X(t, x) \le \lim_{t\to\infty} p^X(t, 0) = 0.$$

Proof. Using the inverse Fourier transform, the heat kernel $p^X(t, \cdot)$ can be written as $p^X(t, x) = (2\pi)^{-d} \int e^{-i\xi x} e^{-t\Phi^X(\xi)} d\xi$. Hence $p(t, 0) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\Phi^X(\xi)} d\xi$ and from the asymptotic behavior of Φ, there exists a constant $c > 0$ such that $\Phi(\xi) \ge c|\xi|^\alpha \ell(|\xi|^2)$ for $|\xi| \ge 1$. Hence $e^{-t\Phi^X(\xi)}$ is integrable in $\mathbb{R}^d$. Now the conclusion follows from the dominated convergence theorem and Lemma 4.1.2. □

The next lemma is similar to [27, Proposition 1.16]. We provide the details for the reader's convenience.

Lemma 4.1.4. *Let $\theta \in (0,1)$ and D be a bounded open set in $\mathbb{R}^d$. Then there exists a $t = t(\theta, |D|)$ such that*

$$\sup_{x \in \mathbb{R}^d} \mathbb{P}_x(\tau_D > nt) \leq \theta^n. \tag{4.1.2}$$

Furthermore if $\widetilde{D}$ is another bounded open set with $|\widetilde{D}| \leq |D|$, then $t(\widetilde{D}) \leq t(D)$, where $t(D)$ represents the constant corresponding to the open set D such that (4.1.2) *holds.*

Proof. For $x \in D$ and any $u > 0$

$$\mathbb{P}_x(\tau_D^X > u) \leq \mathbb{P}_x(X_u \in D) = \int_D p^X(u,x,y)dy \leq p^X(u,0)|D|. \tag{4.1.3}$$

Now using Lemma 4.1.4 take u large enough so that $p^X(u,0)|D| < \theta < 1$. Now from the Markov property of X

$$\mathbb{P}_x(\tau_D^X > (n+1)t) = \mathbb{E}_x\left(\tau_D^X > nt, \mathbb{P}_{X_{nt}}(\tau_D^X > t)\right) \leq \theta \mathbb{E}_x(\tau_D^X > nt) \leq \theta^{n+1}. \tag{4.1.4}$$

Hence by induction we have

$$\mathbb{P}_x(\tau_D^X > nt) \leq \theta^n. \tag{4.1.5}$$

Note that if $\widetilde{D}$ is another bounded open set with $|\widetilde{D}| \leq |D|$, then we can simply take the same u in (4.1.3) so that $p^X(u,0)|\widetilde{D}| \leq p^X(u,0)|D| < \theta$. □

Lemma 4.1.5. *Let D be an open set in $\mathbb{R}^d$. Then there exists a constant $C_1 = C_1(|D|)$ such that*

$$\sup_{x \in D} \mathbb{E}_x \tau_D^X \leq C_1. \tag{4.1.6}$$

Furthermore the same constant C_1 works for all open set $\widetilde{D}$ as long as $|\widetilde{D}| \leq |D|$.

Proof. By an elementary inequality, we have

$$\mathbb{E}_x\left(\frac{\tau_D^X}{t}\right) \leq \sum_{n=0}^{\infty} \mathbb{P}_x\left(\frac{\tau_D^X}{t} > n\right).$$

Hence from (4.1.5) we have

$$\mathbb{E}_x\left(\frac{\tau_D^X}{t}\right) \leq \frac{1}{1-\theta}.$$

Hence we have $\sup_{x\in D} \mathbb{E}_x(\tau_D) \le \frac{t}{1-\theta}$. Note that if $\widetilde{D}$ is another bounded open set with $|\widetilde{D}| \le |D|$, then we can simply take $t(\widetilde{D}) \le t(D)$ in Lemma 4.1.4 and get $\sup_{x\in D} \mathbb{E}_x(\tau^X_{\widetilde{D}}) \le \frac{t(D)}{1-\theta}$. □

The proof of next lemma is identical to [32, Lemma 2.4]. We provide the details for reader's convenience.

Lemma 4.1.6. *Let X be subordinate Brownian motions and Y be perturbations of subordinate Brownian motions with the Lévy density $J^Y(x)$ such that $\sigma := J^Y - J^X$ is integrable on $\mathbb{R}^d$ and D be a bounded open set. Then there exists a constant $C_2 = C_2(|D|, \sigma)$ such that*

$$C_2^{-1}\mathbb{E}^Y_x \tau_D \le \mathbb{E}^X_x \tau_D \le C_2 \mathbb{E}^Y_x \tau_D.$$

Furthermore the same constant C_2 works for all open sets $\widetilde{D}$ as long as $|\widetilde{D}| \le |D|$.

Proof. Suppose that $\sigma = \sigma_+ - \sigma_-$ is the Jordan decomposition of σ. Let V_t be compound Poisson processes independent of X_t with the Lévy measure σ_- and let V_t' be compound Poisson processes independent of Y_t with the Lévy measure σ_+. We put $Z_t = X_t + V_t$. Then, of course, we have $\{Z_t\} = \{Y_t + V_t'\}$ in distribution. Hence it is enough to show that $\mathbb{E}_x \tau^Z_D \asymp \mathbb{E}_x \tau^X_D$.

Let us define a stopping time T by $T = \inf\{t > 0 : V_t \ne 0\}$. The processes X_t and V_t are mutually independent. Therefore X_t and T are independent as well. Besides, $Z_t = X_t$ for $0 \le t < T$. We set $m = \sigma_-(\mathbb{R}^d)$.

First, we claim that $\mathbb{E}_x(\tau^X_D) \le 2\mathbb{E}_x(\tau^X_D \wedge t)$ for t large enough. Indeed, by the Markov property and Lemma 4.1.5 we have

$$\begin{aligned}
\mathbb{E}_x \tau^X_D &= \mathbb{E}_x(\tau^X_D \wedge t) + \mathbb{E}_x\left(\tau^X_D > t; \tau^X_D - t\right) \\
&= \mathbb{E}_x(\tau^X_D \wedge t) + \mathbb{E}_x\left(\tau^X_D > t; \mathbb{E}_{X_t}\tau^X_D\right) \\
&\le \mathbb{E}_x(\tau^X_D \wedge t) + C(D)\mathbb{P}_x(\tau^X_D > t) \\
&\le \mathbb{E}_x(\tau^X_D \wedge t) + C(D)\frac{\mathbb{E}_x\tau^X_D}{t}, \qquad (4.1.7)
\end{aligned}$$

which proves our claim for $t \ge 2C$.

Because $\tau_D^Z \wedge T = \tau_D^X \wedge T$, by the independence of T and X_t we get

$$\begin{aligned} \mathbb{E}_x\tau_D^Z &\geq \mathbb{E}_x(\tau_D^Z \wedge T) = \mathbb{E}_x(\tau_D^X \wedge T) = \int_0^\infty \mathbb{E}_x(\tau_D^X \wedge t) m e^{-mt} dt \\ &\geq \int_{2C}^\infty \mathbb{E}_x(\tau_D^X \wedge t) m e^{-mt} dt \geq \frac{1}{2} e^{-2Cm} \mathbb{E}_x \tau_D^X. \end{aligned} \tag{4.1.8}$$

Now, we prove the upper bound. Again, by the strong Markov property and Lemma 4.1.5 we get

$$\begin{aligned} \mathbb{E}_x\tau_D^Z &= \mathbb{E}_x(\tau_D^Z \wedge T) + \mathbb{E}_x(\tau_D^Z > T; \tau_D^Z - T) \\ &\leq \mathbb{E}_x\tau_D^X + \mathbb{E}_x(\tau_D^Z > T; \mathbb{E}_{Z_T}\tau_D^Z) \\ &\leq \mathbb{E}_x\tau_D^X + C\mathbb{P}_x(\tau_D^Z > T). \end{aligned} \tag{4.1.9}$$

We also have

$$\mathbb{P}_x(\tau_D^Z > T) \leq \mathbb{P}_x(\tau_D^X \geq T) = m \int_0^\infty \mathbb{P}_x(\tau_D^X \geq t) e^{-mt} dt \leq m \mathbb{E}_x \tau_D^X, \tag{4.1.10}$$

which gives

$$\mathbb{E}_x\tau_D^Z \leq (1 + Cm)\mathbb{E}_x\tau_D^X,$$

where $m = \sigma_-(\mathbb{R}^d)$ and $C = C(D)$ is a constant in Lemma 4.1.5 such that $\sup_{x\in D} \mathbb{E}_x\tau_D^X \leq C(D)$.

Now let $c_1 := max(1 + Cm, 2e^{2Cm})^2$ so that

$$c_1^{-1/2}\mathbb{E}_x\tau_D^X \leq \mathbb{E}_x\tau_D^Z \leq c_1^{1/2}\mathbb{E}_x\tau_D^X.$$

Similarly let $c_2 := max(1 + C\tilde{m}, 2e^{2C\tilde{m}})^2$, where $\tilde{m} = \sigma_+(\mathbb{R}^d)$, so that

$$c_2^{-1/2}\mathbb{E}_x\tau_D^Y \leq \mathbb{E}_x\tau_D^Z \leq c_2^{1/2}\mathbb{E}_x\tau_D^Y.$$

Take $C_2 := max(c_1, c_2)$ and this gives

$$\mathbb{E}_x\tau_D^X \leq C_2^{1/2}\mathbb{E}_x\tau_D^Z \leq C_2\mathbb{E}_x\tau_D^Y, \quad \mathbb{E}_x\tau_D^X \geq C_2^{-1/2}\mathbb{E}_x\tau_D^Z \geq C_2^{-1}\mathbb{E}_x\tau_D^Y,$$

which proves the first part of the lemma.

For the second part of the lemma, suppose that $\widetilde{D}$ is a bounded open set in $\mathbb{R}^d$ such that $|\widetilde{D}| \le |D|$. We choose a constant $C = C(D)$ from Lemma 4.1.5 such that

$$\sup_{x\in\widetilde{D}} \mathbb{E}_x^X \tau_{\widetilde{D}} \le C(D).$$

After repeating the same argument in (4.1.7), we have for $t \ge 2C(D)$,

$$\mathbb{E}_x \tau_{\widetilde{D}}^X \le 2\mathbb{E}_x(\tau_{\widetilde{D}}^X \wedge t).$$

Hence as long as $|\widetilde{D}| \le |D|$ we can repeat the same proof in (4.1.8) for the open set $\widetilde{D}$ with the constant $C = C(D)$ and get

$$\mathbb{E}_x \tau_{\widetilde{D}}^Z \ge \frac{1}{2} e^{-2C(D)m} \mathbb{E}_x \tau_{\widetilde{D}}^X.$$

Similarly from (4.1.9)

$$\mathbb{E}_x \tau_{\widetilde{D}}^Z \le \mathbb{E}_x \tau_{\widetilde{D}}^X + C(D)\mathbb{P}_x(\tau_{\widetilde{D}}^Z > T), \tag{4.1.11}$$

$$\mathbb{P}_x(\tau_{\widetilde{D}}^Z > T) \le \mathbb{P}_x(\tau_{\widetilde{D}}^Z \ge T) = m\int_0^\infty \mathbb{P}_x(\tau_{\widetilde{D}}^X \ge t)e^{-mt}dt \le m\mathbb{E}_x\tau_{\widetilde{D}}^X. \tag{4.1.12}$$

Now from (4.1.11), (4.1.12), as long as $|\widetilde{D}| \le |D|$, we have

$$\mathbb{E}_x \tau_{\widetilde{D}}^Z \le (1 + C(D)m)\mathbb{E}_x \tau_{\widetilde{D}}^X.$$

Hence for the same constant $C_2 = max(c_1, c_2)$ where $c_1 := max(1 + C(D)m, 2e^{2C(D)m})^2$ and $c_2 := max(1 + C(D)\tilde{m}, 2e^{2C(D)\tilde{m}})^2$, as long as $|\widetilde{D}| \le |D|$, we have

$$\mathbb{E}_x \tau_{\widetilde{D}}^X \le C_2^{1/2} \mathbb{E}_x \tau_{\widetilde{D}}^Z \le C_2 \mathbb{E}_x \tau_{\widetilde{D}}^Y, \quad \mathbb{E}_x \tau_{\widetilde{D}}^X \ge C_2^{-1/2} \mathbb{E}_x \tau_{\widetilde{D}}^Z \ge C_2^{-1} \mathbb{E}_x \tau_{\widetilde{D}}^Y.$$

□

Remark 4.1.7. *In fact, Lemma 4.1.5 can be stated for more general settings. The lemma can be stated for two pure jump Lévy processes with* $\sigma := J^Y - J^X$ *being integrable on* $\mathbb{R}^d$ *as long as* X *satisfies Lemma 4.1.3.*

Proof of Theorem 4.1.1. We already know from the previous chapter that there is a constant $c = c(D)$ such that (4.1.1) holds for $c(D)$. The only thing to prove is that there are constants C and R such that the constant C is independent of the κ-fat domain D as long as $|D| \le R$. We establish this by proof by contradiction. Suppose that (4.1.1) is false. Then there must exist R, $c_n > 0$, and κ-fat domains D_n such that $\sup_n \{|D_n|\} \le R$ and

$$G^X_{D_n}(x,y) > c_n G^Y_{D_n}(x,y), \tag{4.1.13}$$

where $c_n \to \infty$ as $n \to \infty$. By integrating each side of (4.1.13) in terms of y over D_n, we have

$$\mathbb{E}^X_x \tau_{D_n} \ge c_n \mathbb{E}^Y_x \tau_{D_n}.$$

But since $c_n \to \infty$ as $n \to \infty$, this contradicts Lemma 4.1.6 and this proves the theorem. □

4.2 Harnack Principle

In this section, we will prove a version of the Harnack principle for nonnegative Harmonic functions with respect to processes Y which vanish outside a small ball. We will follow the argument in [37, Chapter 4] closely but since we are only interested in harmonic functions that vanish outside a small ball, the ingredients necessary to prove the Harnack principle is actually less than those in [37].

Now as the first step, from Theorem 4.1.1 there exists a constant C_3 such that for any κ-fat domains D with $|D| \le |B(0, \frac{1}{2})|$, we have

$$C_3^{-1} G^X_D(x,y) \le G^Y_D(x,y) \le C_3 G^X_D(x,y) \quad x, y \in D.$$

The constant C_3 will not change in the rest of the chapter.

We now start with some explanation about the Poisson kernel. It follows from [52, Theorem 1] that for the processes X, Y and for any bounded Lipschitz domains D,

$$\mathbb{P}_x(X_{\tau_D} \in \partial D) = \mathbb{P}_x(Y_{\tau_D} \in \partial D) = 0 \quad , \quad x \in D.$$

Hence from the Ikeda-Watanabe formula, the exit distributions of X and Y are completely determined by their Poisson kernels $K_D^X(x,z)$ and $K_D^Y(x,z)$, respectively. Namely for bounded Lipschitz domains D and $f \geq 0$, it follows that

$$\mathbb{E}_x\left[f(X_{\tau_D})\right] = \int_{\overline{D}^c} K_D^X(x,z)f(z)dz, \quad \mathbb{E}_x\left[f(Y_{\tau_D})\right] = \int_{\overline{D}^c} K_D^Y(x,z)f(z)dz, \quad x \in D \tag{4.2.1}$$

where

$$K_D^X(x,z) = \int_D G_D^X(x,y)J^X(y,z)dy, \quad K_D^Y(x,z) = \int_D G_D^Y(x,y)J^Y(y,z)dy. \tag{4.2.2}$$

Note that without the Lipschitz assumption on D, from the Lévy system of Y, it follows that

$$\mathbb{E}_x\left[f(Y_{\tau_D}), Y_{\tau_D^-} \neq Y_{\tau_D}\right] = \int_{\overline{D}^c} K_D^Y(x,z)f(z)dz, \quad x \in D.$$

Now we estimate the Poisson kernel for a ball $K^Y_{B(x_0,r)}$. Define $A(x,a,b) := \{y \in \mathbb{R}^d : a \leq |y-x| < b\}$ for $0 \leq a < b$.

Lemma 4.2.1. *There exists a constant c_1 such that for all $r \leq \frac{1}{2}$ and $z \in A(x_0,r,1-r)$,*

$$c_1^{-1}K^X_{B(x_0,r)}(x,z) \leq K^Y_{B(x_0,r)}(x,z) \leq c_1 K^X_{B(x_0,r)}(x,z). \tag{4.2.3}$$

Proof. For $y \in B(x_0,r)$ and $z \in A(x_0,r,1-r)$, $|y-z| \leq |y-x_0| + |x_0-z| \leq r + (1-r) = 1$. Hence from (4.0.1), Theorem 4.1.1, and (4.2.2),

$$\begin{aligned} K^Y_{B(x_0,r)}(x,z) &= \int_{B(x_0,r)} G^Y_{B(x_0,r)}(x,y)j^Y(y,z)dz \\ &\leq c\int_{B(x_0,r)} G^X_{B(x_0,r)}(x,y)j^X(y,z)dz = cK^X_{B(x_0,r)}(x,z). \end{aligned}$$

The other direction can be done in a similar way. □

Lemma 4.2.2. *There exists a constant c_1 such that for all $r \leq \frac{1}{2}$, $x_1, x_2 \in B(x_0, \frac{r}{2})$, and $z \in A(x_0,r,1-r)$,*

$$c_1^{-1}K^Y_{B(x_{0,r})}(x_1,z) \leq K^Y_{B(x_{0,r})}(x_2,z) \leq c_1 K^Y_{B(x_{0,r})}(x_1,z). \tag{4.2.4}$$

Proof. It follows from [41, Proposition 4.11] that there exists a constant c_1 such that

$$c_1^{-1} K^X_{B(x_0,r)}(x_1,z) \le K^X_{B(x_0,r)}(x_2,z) \le c_1 K^X_{B(x_0,r)}(x_1,z), \quad x \in B(x_0, r/2), z \in \overline{B(x_0,r)}^c. \tag{4.2.5}$$

Now the result follows from Lemma 4.2.1 and (4.2.5). □

Note that σ is bounded outside the unit ball $B(0,1) \subset \mathbb{R}^d$. Let $M := \sup_{|x|\ge 1} |\sigma(x)|$.

Lemma 4.2.3. *There exists a constant c_1 such that for all $r \le \frac{1}{2}$, $x \in B(x_0, r)$, and $z \in A(x_0, 1-r, 1+\frac{r}{2})$,*

$$K^Y_{B(x_0,r)}(x,z) \le c_1 K^X_{B(x_0,r)}(x,z).$$

Proof. Without losing generality, we may assume $x_0 = 0$. From Theorem 3.4.13, (4.0.1), (4.2.2), and the boundedness of σ outside the unit ball, we have

$$\begin{aligned}
& K^Y_{B(0,r)}(x,z) \\
= & \int_{B(0,r)} G^Y_{B(0,r)}(x,y) J^Y(y,z) dy \\
= & \int_{B(0,r)\cap\{|y-z|\le 1\}} G^Y_{B(0,r)}(x,y) J^Y(y,z) dy + \int_{B(0,r)\cap\{1<|y-z|\le 1+\frac{3r}{2}\}} G^Y_{B(0,r)}(x,y) J^Y(y,z) dy \\
\le & \; c_1 \int_{B(0,r)\cap\{|y-z|\le 1\}} G^X_{B(0,r)}(x,y) J^X(y,z) dy + \int_{B(0,r)\cap\{1<|y-z|\le 1+\frac{r}{2}\}} G^X_{B(0,r)}(x,y) M dy.
\end{aligned} \tag{4.2.6}$$

Since $|y-z| \le |y| + |z| \le r + (1+\frac{r}{2}) \le 1 + \frac{3r}{2} \le \frac{7}{2}$ for $y \in B(0,r)$ and $z \in A(x_0, 1-r, 1+\frac{r}{2})$, we have $J^X(y,z) \ge j^X(\frac{7}{2}) \ge c_2^{-1} M$. Hence it follow that (4.2.6) is bounded above by

$$\begin{aligned}
& c_1 \int_{B(0,r)\cap\{|y-z|\le 1\}} G^X_{B(0,r)}(x,y) J^X(y,z) dy + c_2 \int_{B(0,r)\cap\{1<|y-z|\le 1+\frac{r}{2}\}} G^X_{B(0,r)}(x,y) J^X(y,z) dy, \\
\le & \; c_3 \int_{B(0,r)} G^X_{B(0,r)}(x,y) J^X(y,z) dy = c_3 K^X_{B(0,r)}(x,z).
\end{aligned}$$

□

Now we are ready to prove the main result of this section, which is a version of the Harnack principle for harmonic functions with respect to Y that vanish outside a small ball. Note that the ingredient to prove the Harnack principle in this setting is much less than those that appear in [37] because we only consider harmonic functions that vanish outside a small ball.

Theorem 4.2.4. *There exists a constant c_1 such that for any $r \le \frac{1}{2}$ and a nonnegative regular harmonic function u on $B(x_0, r)$ with respect to Y that vanishes on $B(x_0, 1-r)^c$, we have*

$$c_1^{-1} u(y) \le u(x_0) \le c_1 u(y), \quad y \in B(x_0, r/2). \tag{4.2.7}$$

Proof. It follows from the fact that u is regular harmonic in $B(x_0, r)$, (4.2.1), and Lemma 4.2.2 that for any $y \in B(x_0, r/2)$,

$$\begin{aligned} u(y) &= \mathbb{E}_y[u(Y_{\tau_{B(x_0,r)}})] = \int_{\overline{B(x_0,r)}^c} u(z) K^Y_{B(x_0,r)}(y,z)dz \\ &= \int_{A(x_0,r,1-r)} u(z) K^Y_{B(x_0,r)}(y,z)dz \le c \int_{A(x_0,r,1-r)} u(z) K^Y_{B(x_0,r)}(x_0,z)dz = cu(x_0) \end{aligned}$$

for some constant c. The other inequality can be done in a similar way. □

4.3 Boundary Harnack Principle

In this section we prove a version of the boundary Harnack principle for nonnegative harmonic functions with respect to Y that vanish outside a small ball. We will closely follow the argument in [37, 40, 41]. The main ingredients to prove the boundary Harnack principle is a comparison of harmonic measures (4.3.6), Carleson type estimate (Lemma 4.3.2), and the Harnack principle (Theorem 4.2.4). We begin with the comparison of harmonic measures.

Let $\mathcal{A}$ be the L^2 generator of Y and $C_c^\infty(\mathbb{R}^d)$ be the family of the infinitely differentiable functions on $\mathbb{R}^d$ with compact support. Then it is well known (see [37, page 152-153]) that $C_c^\infty(\mathbb{R}^d) \subseteq Dom(\mathcal{A})$ and for any $\phi \in C_c^\infty(\mathbb{R}^d)$ and $\phi(x) = 0$, we have

$$\mathbb{E}_x[\phi(Y_{\tau_D})] = \int_D G^Y_D(x,y) \mathcal{A}_\alpha \phi(y) dy. \tag{4.3.1}$$

Take a sequence of radial functions $\phi_m \in C_c^\infty(\mathbb{R}^d)$ such that $0 \le \phi_m \le 1$,

$$\phi_m(y) = \begin{cases} 0 & |y| < 1/2 \\ 1 & 1 \le |y| \le m+1 \\ 0 & |y| > m+2 \end{cases}$$

and that $\sum_{i,j} |\frac{\partial^2}{\partial y_i \partial y_j}\phi_m|$ is uniformly bounded. Define $\phi_{m,r}(y) := \phi_m(\frac{y}{r})$. The key step is to show that there is a constant $c = c(d, \alpha, \ell)$ such that for every $\phi_{m,r} \in C_c^\infty(\mathbb{R}^d)$

$$\sup_{M\ge 1} \sup_{y\in\mathbb{R}^d} |\mathcal{A}\phi_{m,r}(y)| \le c\frac{\ell(r^{-2})}{r^\alpha}. \tag{4.3.2}$$

Note that from Theorem 3.1.1 or [40, Lemma 3.10], there is a constant c_1 such that

$$j^X(y) \le c_1 \frac{\ell(|y|^{-2})}{|y|^{d+\alpha}}, \quad |y| \le 1. \tag{4.3.3}$$

Hence it follows from (4.0.1), (4.3.3), boundedness of σ outside the unit ball, and the fact that ϕ_m is bounded, we have

$$\begin{aligned}
& |\mathcal{A}\phi_{m,r}(x)| \\
\le \; & c_1 \left| \int_{\mathbb{R}^d} \left(\phi_{m,r}(x+y) - \phi_{m,r}(x) - (\nabla\phi_{m,r}(x)\cdot y)1_{|x|\le r}(y)\right) j^Y(y)dy \right| \\
\le \; & c_2 \left| \int_{|y|<r} \left(\phi_{m,r}(x+y) - \phi_{m,r}(x) - (\nabla\phi_{m,r}(x)\cdot y)1_{|x|\le r}(y)\right) j^Y(y)dy \right. \\
+ \; & \left. \int_{r\le|y|<1} \left(\phi_{m,r}(x+y) - \phi_{m,r}(x)\right) j^Y(y)dy + \int_{1\le|y|<\infty} \left(\phi_{m,r}(x+y) - \phi_{m,r}(x)\right) j^Y(y)dy \right| \\
\le \; & c_3 \left(\left| \int_{|y|<r} \left(\phi_{m,r}(x+y) - \phi_{m,r}(x) - (\nabla\phi_{m,r}(x)\cdot y)\right) j^X(y)dy \right| + \int_{r\le|y|<1} j^X(y)dy + 1 \right) \\
\le \; & c_4 \left(\frac{1}{r^2} \int_{|y|<r} |y|^2 j^X(y)dy + \int_{r<|y|<1} j^X(y)dy + 1 \right) \\
\le \; & c_5 \left(\frac{1}{r^2} \int_{|y|<r} |y|^2 \frac{\ell(|y|^{-2})}{|y|^{d+\alpha}}dy + \int_{r<|y|<1} \frac{\ell(|y|^{-2})}{|y|^{d+\alpha}}dy + 1 \right) \\
\le \; & c_6 \frac{\ell(r^{-2})}{r^\alpha}.
\end{aligned}$$

Now combining (4.3.1) and (4.3.2), for any $x \in D \cap B(0, r/2)$ we have

$$\mathbb{P}_x(Y_{\tau_D} \in B(0,r)^c) = \lim_{m\to\infty} \mathbb{P}_x\left(Y_{\tau_D} \in A(0,r,(m+1)r)\right) \le c_6 r^{-\alpha} \ell(r^{-2}) \int_D G_D^Y(x,y)dy. \quad (4.3.4)$$

The next lemma is similar to [40, Lemma 4.2] or [41, Lemma 4.16].

Lemma 4.3.1. *There exists a constant $c > 0$ such that for any open set D with $B(A, \kappa r) \subseteq D \subseteq B(0, r)$ for $r \le \frac{1}{2}$ and $\kappa \in (0, 1/2]$, we have for every $x \in D \setminus B(A, \kappa r)$*

$$\int_D G_D^Y(x,y)dy \le c r^\alpha \kappa^{-d-\alpha/2} \frac{1}{\ell((4r)^{-2})} \left(1 + \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((4r)^{-2})}\right) \mathbb{P}_x(Y_{\tau_{D\setminus B(A,\kappa r)}} \in B(A,\kappa r)).$$

Proof. Let $\Omega := D \setminus B(A, \kappa r)$. Note that for $y \in B(A, 1/2\kappa r) \subseteq D$, $|y - z| \le |y| + |z| \le 2r \le 1$. Hence we have

$$K_\Omega^Y(x,y) = \int_\Omega G_\Omega^Y(x,z) J^Y(z,y)dz \ge c_1 K_\Omega^X(x,y)$$

for some constant $c_1 > 0$. Then we have from [40, Lemma 4.2] or [41, Lemma 4.17]

$$\begin{aligned}
&\mathbb{P}_x(Y_{\tau_\Omega} \in B(A,\kappa r)) = \int_{B(A,\kappa r)} K_\Omega^Y(x,y)dy \\
\ge\ & c_1 \int_{B(A,\kappa r)} K_\Omega^X(x,y)dy = c_1 \mathbb{P}_x(X_{\tau_\Omega} \in B(A,\kappa r)) \\
\ge\ & c_2 \left(r^\alpha \kappa^{-d-\alpha/2} \frac{1}{\ell((4r)^{-2})} \left(1 + \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((4r)^{-2})}\right)\right)^{-1} \cdot \int_D G_D^X(x,y)dy \\
\ge\ & c_3 \left(r^\alpha \kappa^{-d-\alpha/2} \frac{1}{\ell((4r)^{-2})} \left(1 + \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((4r)^{-2})}\right)\right)^{-1} \cdot \int_D G_D^Y(x,y)dy. \quad (4.3.5)
\end{aligned}$$

□

Thus from (4.3.4) and (4.3.5) we have proved for every $r \le \frac{1}{2}$

$$\mathbb{P}_x(Y_{\tau_D} \in B(0,r)^c) \le c\kappa^{-d-\alpha/2} \frac{\ell(r^{-2})}{\ell((4r)^{-2})} \left(1 + \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((4r)^{-2})}\right) \mathbb{P}_x(Y_{\tau_{D\setminus B(A,\kappa r)}} \in B(A,\kappa r)), \quad (4.3.6)$$

for some constant $c > 0$ and $x \in D \setminus B(A, \kappa r)$.

Now we focus on proving Carleson type estimate for nonnegative harmonic functions with respect to Y, which is the second ingredient to prove the boundary Harnack principle.

Lemma 4.3.2. *There exists a constant $R_2 \in (0, \frac{1}{4}]$ such that the following property holds. Assume that $B(A, \kappa r) \subset D \cap B(Q, r)$, $r \le R_2$. Suppose that $u(x)$ is a nonnegative regular harmonic function in $B(Q, 2r) \cap D$ and vanishing on $(B(Q, 2r) \cap D^c) \cup B(Q, 1-2r)^c$ and $v(x)$ is a regular harmonic function on $D \cap B(Q, r)$ defined by,*

$$v(x) = \begin{cases} u(x) & \text{on } B(Q, \frac{3r}{2})^c \\ 0 & \text{on } A(Q, r, \frac{3r}{2}) \cup (B(Q, r) \cap D^c). \end{cases}$$

Then there exists a constant $c > 0$ such that

$$u(A) \ge v(A) \ge c\kappa^{\alpha} \frac{\ell((2r)^{-2})}{\ell((\kappa r)^{-2})} u(x), \quad x \in D \cap B(Q, \frac{3r}{2}).$$

Proof. Without losing generality, we may assume $Q = 0$. First, from Lemma 4.2.2, [40, Proposition 3.8], or [41, Proposition 4.10] and from the fact $|y - A| \le |y| + |A| \le 2|y|$ it follows

$$\begin{aligned} v(A) &= \mathbb{E}_A \left[v(Y_{\tau_{D \cap B(0,r)}}) \right] \\ &\ge \mathbb{E}_A \left[v(Y_{\tau_{D \cap B(0,r)}}); Y_{\tau_{D \cap B(0,r)}} \ne Y_{\tau_{(D \cap B(0,r))-}} \right] \\ &= \int_{B(0, \frac{3r}{2})^c} u(y) K^Y_{D \cap B(0,r)}(A, y) dy \\ &\ge \int_{A(0, \frac{3r}{2}, 1-2r)} u(y) K^Y_{B(A, \kappa r)}(A, y) dy \\ &\ge c_1 \int_{A(0, \frac{3r}{2}, 1-2r)} u(y) K^X_{B(A, \kappa r)}(A, y) dy \\ &\ge c_2 \int_{A(0, \frac{3r}{2}, 1-2r)} u(y) j^X(|y - A|) \frac{(\kappa r)^{\alpha}}{\ell((\kappa r)^{-2})} dy, \\ &\ge c_3 \frac{(\kappa r)^{\alpha}}{\ell((\kappa r)^{-2})} \times \int_{A(0, \frac{3r}{2}, 1-2r)} u(y) j^X(|y|) dy. \end{aligned}$$

Hence we have shown that there exists a constant $c_3 > 0$ such that

$$v(A) \ge c_3 \frac{(\kappa r)^{\alpha}}{\ell((\kappa r)^{-2})} \cdot \int_{A(0, \frac{3r}{2}, 1-2r)} u(y) j^X(|y|) dy. \tag{4.3.7}$$

From [40, Equation 4.4] or [41, Equation 4.34], there exists a $\sigma \in (\frac{10}{6}r, \frac{11}{6}r)$ and a constant c_4

such that for any positive function u with respect to Y,

$$\int_{A(0,\sigma,2r)} \ell((|y|-\sigma)^{-2})^{1/2}(|y|-\sigma)^{-\frac{\alpha}{2}}u(y)dy \le c_4 \frac{r^{-\alpha/2}}{\ell((2r)^{-2})^{1/2}} \int_{A(0,\frac{10r}{6},2r)} \ell(|y|^{-2})u(y)dy. \quad (4.3.8)$$

Secondly, we derive an upper bound of $u(x)$. From the harmonicity of u

$$\begin{aligned} u(x) &= \mathbb{E}_x[u(Y_{\tau_{D\cap B(0,\sigma)}})] \\ &= \mathbb{E}_x[u(Y_{\tau_{D\cap B(0,\sigma)}}); Y_{\tau_{D\cap B(0,\sigma)}} \in B(0,\sigma)^c] \\ &= \mathbb{E}_x[u(Y_{\tau_{D\cap B(0,\sigma)}}); Y_{\tau_{D\cap B(0,\sigma)}} \in B(0,\sigma)^c, Y_{\tau_{D\cap B(0,\sigma)}} = Y_{\tau_{B(0,\sigma)}}] \\ &\le \mathbb{E}_x[u(Y_{\tau_{B(0,\sigma)}}); Y_{\tau_{B(0,\sigma)}} \in B(0,\sigma)^c] \\ &= \int_{A(0,\sigma,2r)} u(y)K^Y_{B(0,\sigma)}(x,y)dy + \int_{A(0,2r,1-2r)} u(y)K^Y_{B(0,\sigma)}(x,y)dy. \end{aligned} \quad (4.3.9)$$

Now we estimate the first and second equations of the last expression. For $y \in A(0,2r,1-2r)$, $|y| \le 1-2r \le 1-\sigma$. Hence from [40, Proposition 3.8] or [41, Proposition 4.10] and (4.3.8), we have

$$\begin{aligned} &\int_{A(0,2r,1-2r)} u(y)K^Y_{B(0,\sigma)}(x,y)dy \\ \le\ & c_5 \int_{A(0,2r,1-2r)} u(y)K^X_{B(0,\sigma)}(x,y)dy \\ \le\ & c_6 \int_{A(0,2r,1-2r)} u(y)j^X(|y|-\sigma)\frac{\sigma^{\alpha/2}}{(\ell(\sigma^{-2}))^{1/2}}\frac{(\sigma-|x|)^{\alpha/2}}{\ell((\sigma-|x|)^{-2})^{1/2}}dy \\ \le\ & c_7 \frac{(2r)^\alpha}{\ell((2r)^{-2})} \times \int_{A(0,2r,1-2r)} u(y)j^X(|y|)dy. \end{aligned}$$

In the last inequality we used [40, Lemma 3.11] that there exists a $R \in (0,\frac{1}{4}]$ such that for any $0 < s \le r \le R$, $\frac{s^{\frac{\alpha}{2}}}{\ell((s)^{-2})^{1/2}} \le c\frac{r^{\frac{\alpha}{2}}}{\ell((r)^{-2})^{1/2}}$. Also $|y|-\sigma > \frac{1}{12}|y|$ gives $j^X(|y|-\sigma) < j^X(\frac{1}{12}|y|) \le cj(|y|)$.

For the first term in (4.3.9), from [40, Proposition 3.8] or [41, Proposition 4.10] we get

$$\begin{aligned}
& \int_{A(0,\sigma,2r)} u(y) K^Y_{B(0,\sigma)}(x,y)dy \\
\leq\ & c_8 \int_{A(0,\sigma,2r)} u(y) K^X_{B(0,\sigma)}(x,y)dy \\
\leq\ & c_9 \int_{A(0,\sigma,2r)} u(y) \frac{\sigma^{\alpha/2-d}}{\ell(\sigma^{-2})^{1/2}} \frac{((|y|-\sigma)^{-2})^{1/2}}{(|y|-\sigma)^{\alpha/2}} dy \\
\leq\ & c_{10} r^{-d} \frac{(2r)^{\alpha/2}}{\ell((2r)^{-2})^{1/2}} \int_{A(0,\sigma,2r)} u(y) \frac{((|y|-\sigma)^{-2})^{1/2}}{(|y|-\sigma)^{\alpha/2}} dy \\
\leq\ & c_{11} r^{-d} \frac{(2r)^{\alpha/2}}{\ell((2r)^{-2})^{1/2}} \frac{r^{-\alpha/2}}{\ell((2r)^{-2})^{1/2}} \int_{A(0,\frac{12r}{6},2r)} u(y)\ell(|y|^{-2})dy \\
\leq\ & c_{12} \frac{r^{\alpha}}{\ell((2r)^{-2})} \int_{A(0,\frac{10r}{6},2r)} u(y)\ell(|y|^{-2})|y|^{-d-\alpha}dy \\
\leq\ & c_{13} \frac{r^{\alpha}}{\ell((2r)^{-2})} \int_{A(0,\frac{10r}{6},2r)} u(y) j^X(|y|)dy.
\end{aligned}$$

Combining these two estimates, we have

$$u(x) \leq \frac{r^{\alpha}}{\ell((2r)^{-2})} \int_{A(0,\frac{10r}{6},1-2r)} u(y) j^X(|y|)dy. \tag{4.3.10}$$

From (4.3.7) and (4.3.10), we have

$$v(A) \geq c\kappa^{\alpha} \frac{\ell((2r)^{-2})}{\ell((\kappa r)^{-2})} u(x), \qquad x \in D \cap B(Q, \frac{3r}{2}).$$

□

Now we are ready to prove the main result of this chapter - a version of the boundary Harnack principle for perturbations of subordinate Brownian motions. The proof will be similar to those in [37, 40, 41].

Proof of 4.0.16. Without lose of generality, we may assume $u(A_r(Q)) = v(A_r(Q))$ and $Q = 0$. Since ℓ is slowly varying at ∞, there is a $R_3 \in (0, R_2]$ such that

$$\sup_{r \leq R_3} \left(\frac{\ell((2r)^{-2})}{\ell((\kappa r)^{-2})}, \frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})}, \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((4r)^{-2})}, \frac{\ell(r^{-2})}{\ell((4r)^{-2})} \right) \leq 2. \tag{4.3.11}$$

Now we define regular Harmonic functions $u_1(x)$ and $u_2(x)$ with respect to Y on $B(0,r) \cap D$ as follows.

$$u_1(x) = \begin{cases} 0 & x \in A(0,r,\frac{3r}{2}) \cup (D^c \cup B(0,r)) \\ u(x) & x \in B(0,\frac{3r}{2})^c \end{cases},$$

$$u_2(x) = \begin{cases} 0 & x \in B(0,\frac{3r}{2})^c \\ u(x) & x \in A(0,r,\frac{3r}{2}) \cup (D^c \cup B(0,r)). \end{cases}$$

Clearly $u(x) = u_1(x) + u_2(x)$. First we estimate $u_1(x)$.

$$\begin{aligned} u_1(x) &= \mathbb{E}_x[u_1(Y_{\tau_{D\cap B(0,r)}})] \\ &= \int_{(D\cap B(0,r))^c} u(y) K^Y_{D\cap B(0,r)}(x,y)dy \\ &= \int_{(B(0,\frac{3r}{2}))^c} u(y) K^Y_{D\cap B(0,r)}(x,y)dy \\ &= \int_{(B(0,\frac{3r}{2}))^c \cap B(0,1-2r)} u(y) \int_{D\cap B(0,r)} G^Y_{D\cap B(0,r)}(x,z) J^Y(z,y)dzdy. \end{aligned}$$

For $z \in D\cap B(0,r)$ and $y \in B(0,\frac{3r}{2})^c \cap B(0,1-2r)$, $|z-y| < 1$ and this implies $J^X(z,y) \asymp J^Y(z,y)$. Also $|y-z| \le |y|+|z| \le 3|y|$ and $|y| \le |y-z|+|z| \le 3|y-z|$ and this implies $J^X(z,y) \asymp J^X(0,y) = j^X(|y|)$. Now define,

$$s(x) := \int_{D\cap B(Q,r)} G^Y_{D\cap B(0,r)}(x,z)dz.$$

Then we have

$$c_1^{-1} \int_{B(0,\frac{3r}{2})} u(y) j^X(|y|)dy \le \frac{u_1(x)}{s(x)} \le c_1 \int_{B(0,\frac{3r}{2})} u(y) j^X(|y|)dy,$$

$$\frac{u_1(x)}{s(x)} / \frac{u_1(A)}{s(A)} \le c_2, \quad x \in D \cap B(0,r).$$

By changing the role of $u_1(x)$ and $v_1(x)$ and from Lemma 4.3.2

$$u_1(x) \le c_3 \frac{s(x)}{s(A)} u_1(A) \le c_4 \frac{s(x)}{s(A)} u(A) = c_4 \frac{s(x)}{s(A)} v(A) \le c_5 \frac{v_1(x)}{v_1(A)} v(A) \le c_6 v_1(x) \le c_6 v(x).$$

Hence we now have shown that

$$u_1(x) \le c_6 v(x). \tag{4.3.12}$$

Secondly we estimate $u_2(x)$. From the harmonicity of $u_2(x)$ we have

$$\begin{aligned}
u_2(x) &= \mathbb{E}_x[u_2(Y_{\tau_{D\cap B(Q,r)}})] \\
&= \mathbb{E}_x\left[u(Y_{\tau_{D\cap B(Q,r)}}); Y_{\tau_{D\cap B(Q,r)}} \in A(0,r,\frac{3r}{2})\cap D\right] \\
&\leq \sup_{x\in A(0,r,\frac{3r}{2})\cap D} u(x)\cdot \mathbb{P}_x(Y_{\tau_{D\cap B(Q,r)}} \in B(Q,r)^c) \\
&\leq c_7\kappa^{-\alpha}\frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})}u(A)\mathbb{P}_x(Y_{\tau_{D\cap B(Q,r)}} \in B(Q,r)^c).
\end{aligned}$$

Now from Lemma 4.3.1, (4.3.11) and Theorem 4.2.4, we have

$$\begin{aligned}
u_2(x) &\leq c_8\kappa^{-d-\frac{3\alpha}{2}}\frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})}\frac{\ell(r^{-2})}{\ell((4r)^{-2})}\left(1+\frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((4r)^{-2})}\right)u(A)\mathbb{P}_x\left(Y_{\tau_{D\setminus B(A,\kappa r)}} \in B(A,\kappa r)\right) \\
&\leq c_9 \inf_{x\in B(A,\kappa r)} u(x)\cdot \mathbb{P}_x\left(Y_{\tau_{D\setminus B(A,\kappa r)}} \in B(A,\kappa r)\right) \\
&\leq c_{10}\mathbb{E}_x\left[v(Y_{\tau_{D\setminus B(A,\kappa r)}})\right] \\
&= c_{10}v(x).
\end{aligned}$$

Hence we get

$$u_2(x) \leq c_{10}v(x). \tag{4.3.13}$$

Combining (4.3.12) and (4.3.13), we get

$$u(x) = c_6u_1(x) + c_{10}u_2(x) \leq cv(x), \quad x \in D\cap B(0,\frac{r}{2}).$$

□

Chapter 5

Martin Boundary and Minimal Martin Boundary

In this chapter, we study the Martin boundary of bounded κ-fat domains with respect to Y. In particular, we prove that the Martin boundary and the minimal Martin boundary of bounded κ-fat domains with respect to Y is the same as the Euclidean boundary. One of important ingredients commonly used to prove identifying the Martin and the minimal Martin boundary with the Euclidean boundary is the (scaling invariant) boundary Harnack principle (see [37, 40, 54]). However, it is proved in [38] that the boundary Harnack principle is not true for truncated stable processes in non-convex domains. Therefore we can't use the boundary Harnack principle to identify the Martin and the minimal Martin boundary with the Euclidean boundary when the domain is non-convex. We follow the argument that is close to [38], where the authors proved the similar result about identifying the Martin and the minimal Martin boundary of so called roughly connected κ-fat domains with the Euclidean boundary with respect to truncated stable processes. One of the key ingredient is the uniform Green function comparability result in section 4.1.

5.1 Martin Boundary and Martin Representation

Recall that D is a bounded κ-fat domain with characteristic (r_0, κ) and for each $Q \in \partial D$ and $r \in (0, r_0)$, $A_r(Q)$ is a point in $D \cap B(Q, r)$ satisfying $B(A_r(Q), \kappa r) \subset D \cap B(Q, r)$. Combining the boundary Harnack principle for X (Theorem 3.1.5) and Theorem 3.4.13, we get the following boundary Harnack principle for the Green function $G_D^Y(x, y)$ which will play an important role in this section.

Theorem 5.1.1. *There exists a constant $c = c(D, d, \phi, \rho, \sigma)$ such that for any $Q \in \partial D$, $r \in (0, r_0)$*

and $z, w \in D \setminus B(Q, 2r)$, *we have*

$$c^{-1}\frac{G_D^Y(z, A_r(Q))}{G_D^Y(w, A_r(Q))} \le \frac{G_D^Y(z,x)}{G_D^Y(w,x)} \le c\frac{G_D^Y(z, A_r(Q))}{G_D^Y(w, A_r(Q))}, \quad x \in D \cap B(Q, \frac{r}{2}).$$

The following result was proved in [14] for harmonic functions with respect to stable processes in bounded Lipschitz domains, in [37] for harmonic functions with respect to truncated stable processes in bounded convex domains, and in [38] for harmonic functions with respect to truncated stable processes in roughly connected κ-fat domains. We reproduce the proof for the sake of completeness.

Lemma 5.1.2. *There exist positive constants* $c = c(D, d, \phi, \rho, \sigma)$, $\gamma = \gamma(D, d, \phi, \rho, \sigma) < \alpha$ *and* $R = R(\kappa, \ell)$ *such that for any* $Q \in \partial D$ *and* $r \le R$, *and a nonnegative function* u *which is harmonic with respect to* Y *in* $D \cap B(Q, r)$, *we have*

$$u(A_r(Q)) \le c\left(\frac{2}{\kappa}\right)^{\gamma\kappa}\frac{\ell((\frac{\kappa}{2})^{-2k}r^{-2})}{\ell(r^{-2})}u(A_{(\frac{\kappa}{2})^k r}(Q)).$$

Proof. Let $\eta_k := \left(\frac{\kappa}{2}\right)^k r$, $A_k := A_{\eta_k}(0)$, and $B_k := B(A_k, \eta_{k+1})$. Note that the B_k's are disjoint. Since u is harmonic with respect to Y and all B_k's are disjoint, we have

$$\begin{aligned} u(A_k) &\ge \sum_{l=0}^{k-1}\mathbb{E}_{A_k}\left[u(Y_{\tau_{B_k}}) : Y_{\tau_{B_k}} \in B_l\right] \\ &= \sum_{l=0}^{k-1}\int_{B_l} K_{B_k}^Y(A_k, z)u(z)dz. \end{aligned}$$

From Lemma 4.2.1, the Harnack principle, and [40, Proposition 3.8], we have

$$\begin{aligned} u(A_k) &\ge c_1\sum_{l=0}^{k-1}\int_{B_l} K_{B_k}^Y(A_k, z)u(z)dz \\ &\ge c_2\sum_{l=0}^{k-1}u(A_l)\int_{B_l} K_{B_k}^X(A_k, z)dz \\ &\ge c_3\sum_{l=0}^{k-1}u(A_l)\frac{(\eta_{k+1})^\alpha}{(\eta_{l+1})^\alpha}\frac{\ell(\eta_{l+1})^{-2}}{\ell(\eta_{k+1})^{-2}}. \end{aligned}$$

Hence we have

$$(\eta_k)^{-\alpha}u(A_k)\ell(\eta_{k+1}^{-2}) \geq c_4 \sum_{l=0}^{k-1}(\eta_l)^{-\alpha}u(A_l)\ell(\eta_{l+1}^{-2}).$$

Let $a_k := (\eta_k)^{-\alpha}u(A_k)\ell(\eta_{k+1}^{-2})$. Then $a_k \geq c_4 \sum_{l=1}^{k-1} a_l$. By induction, we have $a_k \geq c_6(1+\frac{c_5}{2})^k a_0$ for some constant $c_5 = c_5(d,\alpha,\ell) > 0$. Thus with $\gamma = \alpha - \ln(1+\frac{c_4}{2})\ln\left(\frac{2}{\kappa}\right)^{-1}$, we get

$$u(A_r(Q)) \leq c_6 \left(\frac{2}{\kappa}\right)^{\gamma k} \frac{\ell\left((\frac{\kappa}{2})^{-2(k+1)}r^{-2}\right)}{\ell\left((\frac{\kappa}{2})^{-2}r^{-2}\right)} u\left(A_{(\frac{\kappa}{2})^k r}(Q)\right).$$

Since ℓ is slowly varying at ∞, we have

$$u(A_r(Q)) \leq c_7 \left(\frac{2}{\kappa}\right)^{\gamma k} \frac{\ell\left((\frac{\kappa}{2})^{-2k}r^{-2}\right)}{\ell(r^{-2})} u\left(A_{(\frac{\kappa}{2})^k r}(Q)\right).$$

□

The next lemma is the analogue of Lemma 4.4 in [38]. Instead of repeating the similar proof in Lemma [38], we use Theorem 3.4.13 and 4.2.3 to make the proof shorter.

Lemma 5.1.3. *There exist positive constants $c_1 = c_1(D,d,\phi,\rho,\sigma)$ and $c_2 = c_2(D,d,\phi,\rho,\sigma) < 1$ such that for any $Q \in \partial D$, $r \in (0,r_0)$, and $w \in D \setminus B(Q,4r)$, we have*

$$\mathbb{E}_x\left[G_D^Y(Y_{\tau^Y_{D\cap B_k}}, w) : Y_{\tau^Y_{D\cap B_k}} \in A(Q,r,1+4^{-k}r)\right] \leq c_1 c_2^k G_D^Y(x,w), \quad x \in D\cap B_k,$$

where $B_k := B(Q,4^{-k}r)$, $k = 0,1,\cdots$.

Proof. It is easy to see by repeating the proof in [40, Lemma 5.4] with slight modifications that

$$\mathbb{E}_x\left[G_D^X(X_{\tau^X_{D\cap B_k}}) :\ X_{\tau^X_{D\cap B_k}} \in A(Q,r,1+\frac{r}{4^k})\right] \leq c_1 c_2^k G_D^X(x,w),$$

for some constants $c_1 > 0$ and $0 < c_2 < 1$. From Theorem 3.4.13, (4.2.1), and Lemma 4.2.3, we have

$$
\begin{aligned}
& \mathbb{E}_x\left[G^Y_D(Y_{\tau^Y_{D\cap B_k}}):\ Y_{\tau^Y_{D\cap B_k}} \in A(Q,r,1+\frac{r}{4^k})\right] \\
= \ & \int_{A(Q,r,1+\frac{r}{4^k})} G^Y_D(y,w)K^Y_{D\cap B_k}(x,y)dy \\
\le \ & c_3\int_{A(Q,r,1+\frac{r}{4^k})} G^X_D(y,w)K^X_{D\cap B_k}(x,y)dy \\
= \ & c_3\mathbb{E}_x\left[G^X_D(X_{\tau^X_{D\cap B_k}}):\ X_{\tau^X_{D\cap B_k}} \in A(Q,r,1+\frac{r}{4^k})\right] \\
\le \ & c_4c_2^kG^X_D(x,w) \\
\le \ & c_5c_2^kG^Y_D(x,w).
\end{aligned}
$$

□

Now the next theorem is about the Hölder continuity of the Martin kernel of Y, which is an analogue of [38, Theorem 4.5] and it follows from Theorem 5.1.1, Lemma 5.1.2, and Lemma 5.1.3 (instead of [14, Lemmas 5, 13, and 14], respectively) in very much the same way as in the case of symmetric stable processes in [14, Lemma 16]. We omit the details since the proof is almost identical to [14, Lemma 16].

Theorem 5.1.4. *There exist positive constants r_1, M_1, c, and η depending on D,d,ϕ,ρ,σ such that for any $Q\in\partial D$, $r<r_1$, and $z\in D\setminus B(Q,M_1r)$, we have*

$$
\left|M^Y_D(z,x)-M^Y_D(z,y)\right| \le c\left(\frac{|x-y|}{r}\right)^{\eta}, \quad x,y\in D\cap B(Q,r).
$$

In particular, the limit $\lim_{D\ni y\to w} M^Y_D(x,y)$ exists for every $w\in\partial D$.

There is a compactification D^M of D, unique upto a homeomorphism, such that $M^Y_D(x,y)$ has a continuous extension to $D\times(D^M\setminus\{x_0\})$ and $M^Y_D(\cdot,z_1)=M^Y_D(\cdot,z_2)$ if and only if $z_1=z_2$ (See, for instance, [46]). The set $\partial^M D=D^M\setminus D$ is called the Martin boundary of D. For $z\in\partial^M D$, set $M^Y_D(\cdot,z)$ to be zero in D^c.

A positive harmonic function u for Y^D is minimal if, whenever v is a positive harmonic function for Y^D with $v\le u$ on D, one must have $u=cv$ for some constant c. The set of points $z\in\partial^M D$ such that $M^Y_D(\cdot,z)$ is minimal harmonic for Y^D is called the minimal Martin boundary of D.

For each $z \in \partial D$ and $x \in D$, let

$$M_D^Y(x,z) := \lim_{D \ni y \to z} M_D^Y(x,y),$$

which exists by Theorem 5.1.4. For each $z \in \partial D$, set $M_D^Y(x,z)$ to be zero for $x \in D^c$.

Lemma 5.1.5. *For every $z \in \partial D$ and $B \subset \overline{B} \subset D$, $M_D^Y(Y_{\tau_B}, z)$ is $\mathbb{P}_x$-integrable.*

Proof. Take a sequence $\{z_m\}_{m\geq 1} \subset D \setminus \overline{B}$ converging to z. Since $M_D^Y(\cdot, z_m)$ is regular harmonic for Y in B, by Fatou's lemma and Theorem 5.1.4,

$$\begin{aligned} \mathbb{E}_x\left[M_D^Y(Y_{\tau_B}, z)\right] &= \mathbb{E}_x\left[\lim_{m\to\infty} M_D^Y(Y_{\tau_B}, z_m)\right] \\ &\leq \liminf_{m\to\infty} M_D^Y(x, z_m) = M_D^Y(x,z) < \infty. \end{aligned}$$

□

Lemma 5.1.6. *For every $z \in \partial D$, $x \in D$, and $0 < r < r_1 \wedge \frac{\delta_D(x)}{3}$,*

$$M_D^Y(x,z) = \mathbb{E}_x\left[M_D^Y\left(Y^D_{\tau_{B(x,y)}}, z\right)\right].$$

Proof. Fix $z \in \partial D$, $x \in D$, and $0 < r < r_1 \wedge \frac{\delta_D(x)}{3}$. Let $\eta_m := \left(\frac{\kappa}{2}\right)^m r$, $z_m := A_{\eta_m}(z)$, $m = 0, 1, \cdots$. Note that for $y \in B(x,r)$ and $w \in B(z_m, \eta_{m+1})$

$$|y-w| \geq |x-w| - |x-y| \geq |x-w| - r \geq |x-z| - |z-w| - r \geq \delta_D(x) - \eta_m - r \geq r. \quad (5.1.1)$$

Hence, $B(z_m, \eta_{m+1}) \subset D \setminus B(x,r)$ for all $m \geq 0$. Thus by the harmonicity of $M_D^Y(\cdot, z_m)$, we have

$$M_D^Y(x, z_m) = \mathbb{E}_x\left[M_D^Y\left(Y_{\tau_{B(x,r)}}, z_m\right)\right].$$

On the other hand, by Theorem 5.1.1, there exist constants $m_0 \geq 0$ and $c_1 > 0$ such that for every $w \in D \setminus B(z, \eta_m)$ and $y \in D \cap B(z, \eta_{m+1})$,

$$M_D^Y(w, z_m) \leq \frac{G_D^Y(w, z_m)}{G_D^Y(x_0, z_m)} \leq c_1 \frac{G_D^Y(w,y)}{G_D^Y(x_0,y)} = c_1 M_D^Y(w,y), \quad m \geq m_0.$$

Let $y \to z \in \partial D$ we get

$$M_D^Y(w, z_m) \leq c_1 M_D^Y(w, z), \quad m \geq m_0, \tag{5.1.2}$$

for every $w \in D \setminus B(z, \eta_m)$.

Hence in order to prove the lemma, it is enough to show that $\{M_D^Y(\cdot, z_m) : m \geq m_0\}$ is $\mathbb{P}_x$-uniformly integrable. Since $M_D^Y(Y_{\tau_{B(x,r)}}, z)$ is $\mathbb{P}_x$-integrable by Lemma 5.1.5, for any $\varepsilon > 0$ there is $N_0 > 1$ such that

$$\mathbb{E}_x \left[M_D^Y \left(Y_{\tau_{B(x,r)}}, z \right) : M_D^Y \left(Y_{\tau_{B(x,r)}}, z \right) > \frac{N_0}{c_1} \right] < \frac{\varepsilon}{2c_1}. \tag{5.1.3}$$

Now by (5.1.2) and (5.1.3)

$$\begin{aligned} & \mathbb{E}_x \left[M_D^Y(Y_{\tau_{B(x,r)}}, z_m) : Y_{\tau_{B(x,r)}} \in D \setminus B(z, \eta_m), M_D^Y(Y_{\tau_{B(x,r)}}, z_m) > N_0 \right] \\ \leq \ & \mathbb{E}_x \left[c_1 M_D^Y(Y_{\tau_{B(x,r)}}, z) : c_1 M_D^Y(Y_{\tau_{B(x,r)}}, z) > N_0 \right] \\ \leq \ & c_1 \frac{\varepsilon}{2c_1} = \frac{\varepsilon}{2}. \end{aligned}$$

Now from (4.2.1) we have

$$\mathbb{E}_x \left[M_D^Y(Y_{\tau_{B(x,r)}}, z_m) : Y_{\tau_{B(x,r)}} \in B(z, \eta_m) \right] = \int_{B(z,\eta_m) \cap D} M_D^Y(w, z_m) K_{B(x,r)}^Y(x, w) dw.$$

For $y \in B(x, r)$ and $w \in B(z, \eta_m) \cap D$, it follows from (5.1.1) that $J^Y(y, w) \leq c_2$ for some constant $c_2 = c_2(r)$, where $c_2(r) = \sup_{|z|>r} J^Y(z) < \infty$. Hence we have from (4.2.2) that

$$\begin{aligned} K_{B(x,r)}^Y(x, w) &= \int_{B(x,r)} G_{B(x,r)}^Y(x, y) J^Y(y, w) dy \\ &\leq c_2(r) \int_{B(x,r)} G_{B(x,r)}^Y(x, y) \\ &\leq c_3(r). \end{aligned}$$

Now we see that

$$\begin{aligned}
& \mathbb{E}_x\left[M_D^Y(Y^D_{\tau_{B(x,r)}}, z_m) : Y_{\tau_{B(x,r)}} \in D\cap B(z,\eta_m)\right] \\
\leq\ & c_3 \int_{B(z,\eta_m)} M_D^Y(w,z_m)dw \\
=\ & c_3 \frac{1}{G_D^Y(x_0,z_m)} \int_{B(z,\eta_m)} G_D^Y(w,z_m)dw.
\end{aligned}$$

Note that by Lemma 5.1.2, there exist $c_4 = c_4(D,\alpha,\ell,m_0)$, $c_5 = c_5(D,\alpha,\ell,m_0,r) > 0$, and $\gamma < \alpha$ such that

$$\begin{aligned}
G_D^Y(x,z_m)^{-1} &\leq c_4(\frac{\kappa}{2})^{-\gamma m} \frac{\ell\left((\frac{\kappa}{2})^{-2(m+1)}(\frac{\kappa}{2})^{-2m_0}r^{-2}\right)}{\ell\left((\frac{\kappa}{2})^{-2}(\frac{\kappa}{2})^{-2m_0}r^{-2}\right)} G_D^Y(x_0,z_m)^{-1} \\
&\leq c_5(\frac{\kappa}{2})^{-\gamma m} \ell\left((\frac{\kappa}{2})^{-2m}(\frac{\kappa}{2})^{-2(m_0+1)}r^{-2}\right).
\end{aligned}$$

On the other hand, by Theorem 3.1.1 and Theorem 3.4.13

$$\begin{aligned}
\int_{B(z,\eta_m)} G_D^Y(w,z_m) &\leq c_6 \int_{B(z,2\eta_m)} \frac{dw}{\ell(|w-z_m|^{-2})|w-z_m|^{d-\alpha}} \\
&\leq c_7 \int_0^{2\eta_m} \frac{s^{\alpha-1}}{\ell(s^{-2})} ds \leq c_8 \frac{(\eta_m)^\alpha}{\ell((2\eta_m)^{-2})}.
\end{aligned}$$

In the last inequality above, we have used [40, Equation (3.16)]. Now it follows from above that there exists $c_8 = c_8(D,\alpha,\ell,m_0,r)$ such that

$$\begin{aligned}
& \mathbb{E}_x\left[M_D^Y(Y^D_{\tau_{B(x,r)}}, z_m) : Y_{\tau_{B(x,r)}} \in D\cap B(z,\frac{2r}{m})\right] \\
\leq\ & c_8(\frac{\kappa}{2})^{(\alpha-\gamma)m} \frac{\ell\left((\frac{\kappa}{2})^{-2m}(\frac{\kappa}{2})^{-2(m_0+1)}r^{-2}\right)}{\ell(\frac{\kappa}{2})^{-2m}(2r)^{-2}}.
\end{aligned}$$

Since ℓ is slowly varying at ∞, we can take $N = N(\varepsilon, D, m_0, r)$ so that for $m > N$

$$\begin{aligned} & \mathbb{E}_x\left[M_D^Y(Y_{\tau_{B(x,r)}} z_m) : M_D^Y(Y_{\tau_{B(x,r)}} z_m) > N\right] \\ \leq\ & \mathbb{E}_x\left[M_D^Y(Y_{\tau_{B(x,r)}} z_m) : Y_{\tau_{B(x,r)}} \in D \cap B(z, \frac{2r}{m})\right] + \\ & \mathbb{E}_x\left[M_D^Y(Y_{\tau_{B(x,r)}} z_m) : M_D^Y(Y_{\tau_{B(x,r)}} z_m) > N, Y_{\tau_{B(x,r)}} \in D \setminus B(z, \frac{2r}{m})\right] \\ \leq\ & c_9(\frac{\kappa}{2})^{(\alpha-\gamma)m} \frac{\ell\left((\frac{\kappa}{2})^{-2m}(\frac{\kappa}{2})^{-2(m_0+1)} r^{-2}\right)}{\ell(\frac{\kappa}{2})^{-2m}(2r)^{-2}} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

As each $M_D^Y\left(Y_{\tau_{B(x,r)}}, z_m\right)$ is $\mathbb{P}_x$-integrable, we conclude that $\{M_D^Y\left(Y_{\tau_{B(x,r)}}, z_m\right) : m \geq m_0\}$ is uniformly integrable under $\mathbb{P}_x$. □

It is easy to see that $\mathbb{P}_x\left(Y_{\tau_U} \in \partial U\right) = 0$ for every smooth open set U ([55, Theorem 1]). Hence, one can follow the proof of [23, Theorem 2.2] or the proof of [38, Theorem 4.8] and show that the two lemmas above imply that $M_D^Y(\cdot, z)$ is harmonic for Y. We omit the details since the proof is almost identical.

Theorem 5.1.7. *For every $z \in \partial D$, the functions $x \mapsto M_D^Y(\cdot, z)$ is harmonic in D with respect to Y.*

Recall that a point $z \in \partial D$ is said to be a regular boundary point for Y if $\mathbb{P}_z(\tau_D^Y = 0) = 1$ and an irregular boundary point if $\mathbb{P}_z(\tau_D^Y = 0) = 0$. It is well known that if $z \in \partial D$ is regular for Y, then for any $x \in D$, $G_D^Y(x, y) \to 0$ as $y \to z$.

Lemma 5.1.8. *(1) If $z, w \in \partial D$, $z \neq w$ and w is a regular boundary point for Y, then $M_D^Y(x, z) \to 0$ as $x \to w$.*

(2) The mapping $(x, z) \mapsto M_D^Y(x, z)$ is continuous on $D \times \partial D$.

Proof. For any $y \in B(Q, r)$, where $r < \left(r_0 \wedge \delta_D(x)\right)/2$, we have from Theorem 5.1.1

$$M_D^Y(x, y) = \frac{G_D^Y(x, y)}{G_D^Y(x_0, y)} \leq c \frac{G_D^Y(x, A_r(Q))}{G_D^Y(x_0, A_r(Q))}, \tag{5.1.4}$$

for some constant $c > 0$. By letting $y \to Q$ in (5.1.4), we have

$$M_D^Y(x, Q) \le c \frac{G_D^Y(x, A_r(Q))}{G_D^Y(x_0, A_r(Q))}.$$

Now the left hand side converges to zero as x approaches to any regular boundary point other than Q.

For the second part of the lemma, note that $M_D^Y(\cdot, Q)$ is harmonic in D and therefore continuous there (see, for example, (4.2.1)). Now we have from Theorem 5.1.4

$$\begin{aligned} |M_D^Y(x,P) - M_D^Y(y,Q)| &\le |M_D^Y(x,P) - M_D^Y(y,P)| + |M_D^Y(y,P) - M_D^Y(y,Q)| \\ &\le |M_D^Y(x,P) - M_D^Y(y,P)| + c\left(\frac{|P-Q|^\eta}{r}\right). \end{aligned}$$

Now the second part of the lemma follows by letting $x \to y$ and $P \to Q$. □

So far we have shown that the Martin boundary of D can be identified with a subset of the Euclidean boundary ∂D. In order to prove that the Martin boundary and the minimal Martin boundary can be identified with the Euclidean boundary, we need some lemmas. The need for these lemmas comes from the existence of irregular boundary points and we will follow arguments that are close to those in [40] and [46].

Lemma 5.1.9. *Suppose that h is a bounded singular harmonic function with respect to Y in a bounded open set D. If there is a set N of zero capacity such that for any $z \in \partial D \setminus N$,*

$$\lim_{D \ni x \to z} h(x) = 0,$$

then h is identically zero.

Proof. The proof is identical to [40, Lemma 5.10] and we provide the details for the reader's convenience. Take an increasing sequences of open sets $\{D_m\}_{m\ge1}$ satisfying $\overline{D_m} \subset D_{m+1}$ and $\cup_{m=1}^\infty D_m = D$. Set $\tau_m = \tau_{D_m}$. Then $\tau_m \uparrow \tau_D$ and $\lim_{m\to\infty} Y_{\tau_m} = Y_{\tau_D}$ by the quasi-left continuity of X. Since N has zero capacity, we have

$$\mathbb{P}_x(Y_{\tau_D} \in N) = 0, \quad x \in D.$$

Therefore by the bounded convergence theorem we have for any $x \in D$,

$$\begin{aligned} h(x) &= \lim_{m\to\infty} \mathbb{E}_x \left(h(Y_{\tau_m}) : \tau_m < \tau_D \right) \\ &= \lim_{m\to\infty} \mathbb{E}_x \left(h(Y_{\tau_m}) 1_{\partial D \setminus N}(Y_{\tau_D}) : \tau_m < \tau_D \right) \\ &= 0. \end{aligned}$$

□

Lemma 5.1.10. *Let D be a bounded κ-fat open set and I be irregular boundary points of D with respect to Y. Then $cap(I) = 0$.*

Proof. By [9, Proposition $II.3.3$], I is semi-polar and it is polar by [30, Theorem 4.1.2]. Hence it follows that $cap(I) = 0$. □

Now we are ready to prove the main theorem of this chapter.

Theorem 5.1.11. *The Martin boundary and the minimal Martin boundary of D with respect to Y can be identified with the Euclidean boundary of D.*

Proof. So far, we have shown that the Martin boundary can be identified with a subset of the Euclidean boundary. We will show that for $z_1, z_2 \in \partial D$, $z_1 \neq z_2$, we have $M^Y_D(\cdot, z_1) \neq M^Y(\cdot, z_2)$ and this will show that the Martin boundary coincides with the Euclidean boundary of D. Let I be the set of irregular points of D with respect to Y. By Lemma 5.1.10, we have $cap(I) = 0$. Take a decreasing sequence of open sets Δ_m containing I such that

$$\lim_{m\to\infty} cap(\Delta_m) = 0.$$

Then we have

$$\lim_{m\to\infty} \mathbb{P}_x(T_{\Delta_m} < \infty) = 0, \quad x \notin \cap_{m=1}^{\infty} \Delta_m.$$

Define $D_k := \{x \in D : dist(x, D^c) > \frac{1}{k}\}$ and ω^x_A be a harmonic measure of A starting at x, that is $\omega^x_A(\cdot) := \mathbb{P}_x(Y_{\tau_A} \in \cdot)$. Without lose of generality, we may assume $x_0 \in D_1 \setminus \overline{\Delta_1}^c$. Hence we have

$$\omega^{x_0}_{D_k}\left(\Delta_m \cap D^c_k\right) = \mathbb{P}_{x_0}\left(Y_{\tau_{D_k}} \in \Delta_m\right),$$

$$\lim_{m\to\infty} \sup_k \omega^{x_0}_{D_k}(\Delta_m \cap D^c_k) = 0.$$

For each $z \in \partial D$, define

$$\nu^z_k(dy) := M^Y_D(y,z)\omega^{x_0}_{D_k}(dy).$$

Then we have $\nu^z_k(\mathbb{R}^d) = \int_{\mathbb{R}^d} M^Y_D(y,z)\omega^{x_0}_{D_k}(dy) = M^Y_D(x_0,z) = 1$. Next we will show that ν^z_k converges weakly to δ_z, the point measure on z, as $k \to \infty$. For given ϵ, let $B := B(z,\varepsilon)$. Then we have

$$\begin{aligned}
\nu^z_k(B^c) &= \int_{B^c} M^Y_D(y,z)\omega^{x_0}_{D_k}(dy) \\
&= \int_{(B^c\cap\Delta_m)\cap D^c_k} M^Y_D(y,z)\omega^{x_0}_{D_k}(dy) + \int_{(B^c\setminus\Delta_m)\cap D^c_k} M^Y_D(y,z)\omega^{x_0}_{D_k}(dy) \\
&\le \sup_{y\in B^c} M^Y_D(y,z)\omega^{x_0}_{D_k}(\Delta_m \cap D^c_k) + \sup_{y\in(B^c\setminus\Delta_m)\cap D^c_k} M^Y_D(y,z).
\end{aligned}$$

By Theorem 5.1.1, $M^Y_D(\cdot,z)$ is bounded on $\cdot \in B^c$. Hence for given ε choose $m = m(\varepsilon)$ such that

$$\omega^{x_0}_{D_k}(\Delta_m \cap D^c_k) < \frac{\varepsilon}{2\sup_{y\in B^c} M^Y_D(y,z)}.$$

Now from Lemma 5.1.8 we can choose $k = k(\varepsilon, m)$ such that $\sup\limits_{y\in(B^c\setminus\Delta_m)\cap D^c_k} M^Y_D(y,z) < \varepsilon/2$. Hence we have shown that $\nu^z_k \Rightarrow \delta_z$ as $k \to \infty$. Hence if $M^Y_D(\cdot,z_1) = M^Y_D(\cdot,z_2)$, we must have $z_1 = z_2$.

Now we will focus on proving the minimal Martin boundary can be identified with the Euclidean boundary. Fix $z \in \partial D$ and suppose that $h \le M^Y_D(\cdot,z)$, where h is nonnegative and harmonic with respect to Y in D. Then there is a finite measure μ on ∂D such that

$$h(\cdot) = \int_{\partial D} M^Y_D(\cdot,w)\mu(dw).$$

If μ is not a multiple of δ_z, then there is a positive measure $\nu \le \mu$ such that $dist(z, supp(\nu)) > 0$. Let

$$u(\cdot) := \int_{\partial D} M^Y_D(\cdot,w)\nu(dw).$$

Then u is a positive harmonic function with respect to Y in D and is bounded above by $M^Y_D(\cdot,z)$. Take $\varepsilon = \frac{1}{2}dist(z, supp(\nu))$. Then by the boundary Harnack principle for the Green function

(Theorem 5.1.1), $M_D^Y(\cdot, z)$ is bounded on $B(z,\varepsilon)^c$ and so is u. Again from the boundary Harnack principle we see that $M_D^Y(\cdot,\cdot)$ is bounded on $B(z,\varepsilon) \times supp(\nu)$, so u is also bounded on $B(z,\varepsilon)$. Since $M_D^Y(x,z) \to 0$ as x approaches any regular boundary points different from z, $u(x) \to 0$ as x approaches any regular boundary points different from z. From Lemma 5.1.9 we see that u is identically zero. Therefore $\mu = c\delta_z$ for some $c > 0$ and $M_D^Y(\cdot, z)$ is minimal harmonic with respect to Y on D. □

As a consequence of Theorem 5.1.11, we conclude that for every nonnegative harmonic function h for Y^D, there exists a unique finite measure μ supported on ∂D such that

$$h(x) = \int_{\partial D} M_D^Y(x,z)\mu(dz), \quad x \in D.$$

We call μ the Martin measure of h.

Furthermore, from Corollary 3.4.15, we get the following sharp estimates on Martin kernels for bounded $C^{1,1}$ domains.

Theorem 5.1.12. *If D is a bounded $C^{1,1}$ domain, there exists $c = c(x_0, D, d, \phi, \rho, \sigma)$ such that*

$$c^{-1}\frac{1}{\sqrt{\phi(\delta_D(x)^{-2})}|x-z|^d} \le M_D^Y(x,z) \le c\frac{1}{\sqrt{\phi(\delta_D(x)^{-2})}|x-z|^d}, \quad x \in D, z \in \partial D.$$

Chapter 6

Trace Estimate of Relativistic Stable Processes

6.1 Introduction and Statement of the Main Results

For any $m > 0$ and $\alpha \in (0, 2)$, a relativistic α-stable process X^m on $\mathbb{R}^d$ with mass m is a Lévy process with characteristic function given by

$$\mathbb{E}\left[\exp(i\xi \cdot (X_t^m - X_0^m))\right] = \exp(-t((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m)), \quad \xi \in \mathbb{R}^d. \tag{6.1.1}$$

The limiting case X^0, corresponding to $m = 0$, is a (rotationally) symmetric α-stable process on $\mathbb{R}^d$ which we will simply denote as X. The infinitesimal generator of X^m is $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$. Note that when $m = 1$, this infinitesimal generator reduces to $1 - (1 - \Delta)^{\alpha/2}$. Thus the 1-resolvent kernel of the relativistic α-stable process X^1 on $\mathbb{R}^d$ is just the Bessel potential kernel. When $\alpha = 1$, the infinitesimal generator reduces to the so-called free relativistic Hamiltonian $m - \sqrt{-\Delta + m^2}$. The operator $m - \sqrt{-\Delta + m^2}$ is very important in mathematical physics due to its application to relativistic quantum mechanics.

In this chapter, we will be interested in the asymptotic behavior of the trace of the semigroup associated with killed relativistic α-stable processes in open sets of $\mathbb{R}^d$. The process X^m has a transition density $p^m(t, x, y) = p^m(t, y - x)$ given by the inverse Fourier transform

$$p^m(t, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi x} e^{-t(|\xi|^2 + m^{2/\alpha})^{\alpha/2} + mt} d\xi.$$

For any open set D in $\mathbb{R}^d$, the killed relativistic α-stable process $X_t^{m,D}$ is defined by

$$X_t^{m,D} = \begin{cases} X_t^m & \text{if } t < \tau_D^m, \\ \partial & \text{if } t \geq \tau_D^m, \end{cases}$$

where $\tau_D^m = \inf\{t > 0 : X_t^m \notin D\}$ is the first exit time of X^m from D. The process $X_t^{m,D}$ is a strong Markov process with a transition density $p_D^m(t, x, y)$ given by

$$p_D^m(t, x, y) = p^m(t, x, y) - r_D^m(t, x, y),$$

with

$$r_D^m(t, x, y) = \mathbb{E}_x \left[t > \tau_D^m ; p^m(t - \tau_D^m, X_{\tau_D^m}^m, y) \right].$$

We denote by $(P_t^{m,D} : t \geq 0)$ the semigroup of X_t^m on $L^2(D)$: for any $f \in L^2(D)$,

$$P_t^{m,D} f(x) := \mathbb{E}_x \left[f(X_t^{m,D}) \right] = \int_D f(y) p_D^m(t, x, y) dy.$$

Whenever D is of finite volume, $P_t^{m,D}$ is a Hilbert-Schmidt operator mapping $L^2(D)$ into $L^\infty(D)$ for every $t > 0$. By general operator theory, there exist an orthonormal basis of eigenfunctions $\{\phi_n^{(m)}\}_{n=1}^\infty$ for $L^2(D)$ and corresponding eigenvalues $\{\lambda_n^{(m)}\}_{n=1}^\infty$ of the generator of the semigroup $P_D^{m,D}$ satisfying

$$0 < \lambda_1^{(m)} < \lambda_2^{(m)} \leq \lambda_3^{(m)} \leq \cdots$$

with $\lambda_n^{(m)} \to \infty$. By definition, we have

$$P_t^{m,D} \phi_n^{(m)}(x) = e^{-\lambda_n^{(m)} t} \phi_n^{(m)}(x), \quad x \in D, t > 0.$$

We also have

$$p_D^m(t, x, y) = \sum_{n=1}^\infty e^{-\lambda_n^{(m)} t} \phi_n^{(m)}(x) \phi_n^{(m)}(y).$$

$\lambda_n^{(0)}$ will be simply denoted by λ_n.

In the remainder of this chapter, we assume $d \geq 2$. We are interested in finding the asymptotic behavior, as $t \to 0$, of the trace defined by

$$Z_D^m(t) = \int_D p_D^m(t,x,x)dx = \sum_{n=1}^{\infty} e^{-\lambda_n^{(m)}t} \int_D (\phi_n^{(m)})^2(x)dx = \sum_{n=1}^{\infty} e^{-\lambda_n^{(m)}t}.$$

It is shown in [3] that for any open set D of finite volume, it holds that

$$\lim_{t\to 0} t^{d/\alpha} Z_D^0 = C_1|D|, \qquad C_1 = \frac{\omega_D \Gamma(d/\alpha)}{(2\pi)^d \alpha}. \tag{6.1.2}$$

This is closely related to the growth of the eigenvalues of $P_t^{0,D}$: if $N^0(\lambda)$ is the number of eigenvalues λ_j such that $\lambda_j \le \lambda$, then it follows from the classical Karamata Tauberian theorem (see for example [29]) that

$$N^0(\lambda) \sim \frac{C_1|D|}{\Gamma(d/\alpha+1)} \lambda^{d/\alpha}, \quad \text{as } \lambda \to \infty. \tag{6.1.3}$$

This is the analogue for killed stable processes of the celebrated Weyl's asymptotic formula for the eigenvalues of the Dirichlet Laplacian. We will see later in this chapter that exactly the same formula is true for relativistic stable processes. That is, the first term in the expansion of $Z_D^m(t)$ is the same as that of $Z_D^0(t)$ and (6.1.3) is also true for relativistic stable processes.

Our main goal in this chapter is to get the asymptotic expansion of $Z_D^m(t)$ as $t \to 0$ under some additional assumptions on the smoothness of the boundary of D. Our work is inspired by the paper [14] for Brownian motion and the papers [3, 4] for stable processes. The first theorem is an asymptotic expansion of $Z_D^m(t)$ with error bound of order $t^{2/\alpha}t^{-d/\alpha}$ in $C^{1,1}$ open sets. To state the result precisely, we need some definitions. Recall that an open set D in $\mathbb{R}^d$ is said to be a (uniform) $C^{1,1}$ open set if there are (localization radius) $R > 0$ and Λ_0 such that for every $z \in \partial D$, there exist a $C^{1,1}$ function $\phi = \phi_z : \mathbb{R}^d \to \mathbb{R}$ satisfying $\phi(0, \cdots, 0) = 0$, $\nabla\phi(0) = (0, \dots, 0)$, $|\nabla\phi(x) - \nabla\phi(y)| \le \Lambda_0|x - z|$ and an orthonormal coordinate system $CS_z : y = (y_1, \cdots, y_{d-1}, y_d) := (\tilde{y}, y_d)$ with origin at z such that $B(z,R) \cap D = \{y = (\tilde{y}, y_d) \in B(0,R) \text{ in } CS_z : y_d > \phi(\tilde{y})\}$. For $x \in \mathbb{R}^d$, let $\delta_D(x)$ denote the Euclidean distance between x and D^c and $\delta_{\partial D}(x)$ the Euclidean distance between x and ∂D. It is well known that a $C^{1,1}$ open set D satisfies both the *uniform interior ball condition* and the *uniform exterior ball condition*: there exists $r_0 < R$ such that for every $x \in D$ with $\delta_{\partial D}(x) < r_0$ and $y \in \mathbb{R}^d \setminus \bar{D}$ with $\delta_{\partial D}(y) < r_0$, there are $z_x, z_y \in \partial D$ so that $|x - z_x| = \delta_{\partial D}(x)$,

$|y-z_y| = \delta_{\partial D}(y)$ and that $B(x_0, r_0) \subset D$ and $B(y_0, r_0) \subset \mathbb{R}^d \setminus \bar{D}$, where $x_0 = z_x + r_0(x-z_x)/|x-z_x|$ and $y_0 = z_y + r_0(y-z_y)/|y-z_y|$. In fact, D is a $C^{1,1}$ open set if and only if D satisfies the uniform interior ball condition and the uniform exterior ball condition (see [1, Lemma 2.2]). In this chapter we call the pair (r_0, Λ_0) the characteristics of the $C^{1,1}$ open set D. For any open set D in $\mathbb{R}^d$, we use $|D|$ to denote the d-dimensional Lebesgue measure of D and $\mathcal{H}^{d-1}(\partial D)$ to denote the $(d-1)$-dimensional Hausdorff measure of ∂D. When D is a $C^{1,1}$ open set, $\mathcal{H}^{d-1}(\partial D)$ is equal to the surface measure $|\partial D|$ of ∂D. We will use H to denote the half space $\{x = (x_1, x_2, \cdots, x_d) : x_1 > 0\}$.

The following is the the first main result of this chapter.

Theorem 6.1.1. *Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. Let k be the largest integer such that $k < \frac{2}{\alpha}$. Then the trace $Z_D^m(t)$ admits the following expansion*

$$Z_D^m(t) = C_1|D|t^{-\frac{d}{\alpha}} - C_2|\partial D|t^{\frac{1-d}{\alpha}} + \frac{\omega_d \Gamma(d/\alpha)|D|}{(2\pi)^d \alpha} t^{-\frac{d}{\alpha}} \sum_{n=1}^{k} \frac{m^n}{n!} t^n + O(\frac{t^{2/\alpha}}{t^{d/\alpha}}),$$

where C_1 is given in (6.1.2) *and*

$$C_2 = \int_0^\infty r_H^0(1, (r, \tilde{0}), (r, \tilde{0}))dr.$$

The second main result of the chapter is an asymptotic expansion of $Z_D^m(t)$ with error bound of order $t^{1/\alpha}t^{-d/\alpha}$ in Lipschitz open sets. Before we state the second main result, we recall the definition of Lipschitz open sets. An open set D in $\mathbb{R}^d$ is called a Lipschitz open set if there exist constants R_0 (localization radius) and $\lambda > 0$ (Lipschitz constant) such that for every $z \in \partial D$ there exist a Lipschitz function $F : \mathbb{R}^{d-1} \to \mathbb{R}$ with Lipschitz constant λ and an orthornormal coordinate system $y = (y_1, \cdots, y_d)$ such that $D \cap B(z, R_0) = \{y : y_d > F(y_1, \cdots, y_{d-1})\} \cap B(z, R_0)$. Here is the second main result.

Theorem 6.1.2. *Suppose that D is a bounded Lipschitz open set in $\mathbb{R}^d$. Let j be the largest integer such that $j \le \frac{1}{\alpha}$. Then the trace $Z_D^m(t)$ admits the following expansion*

$$t^{d/\alpha} Z_D^m(t) = C_1|D| - C_2\mathcal{H}^{d-1}(\partial D)t^{1/\alpha} + \frac{\omega_d \Gamma(d/\alpha)|D|}{(2\pi)^d \alpha} \sum_{n=1}^{j} \frac{m^n}{n!} t^n + o(t^{1/\alpha}),$$

where C_1 and C_2 are the same as in Theorem 6.1.1.

Remark 6.1.3. *Note that the first term in the expansion of $Z_D^m(t)$ is exactly the same as in the case of $Z_D^0(t)$. However the rest of the terms are quite different. We note here that the coefficient of the term of order $t^{1/\alpha}t^{-d/\alpha}$ is the same in the stable process case, but in the case of relativistic stable processes for $C^{1,1,}$ open sets, there are k intermediate terms of the form $t^k t^{-d/\alpha}$, where k is a positive integer less than $2/\alpha$. Since $0 < \alpha < 2$, there is at least one more term involved in the asymptotic expansion of $Z_D^m(t)$ than that of $Z_D^0(t)$ up to order of $t^{2/\alpha}t^{-d/\alpha}$. For Lipschitz open sets, when $\alpha \leq 1$ there are j intermediate terms of the form $t^j t^{-d/\alpha}$, where j is an integer that is less than or equal to $1/\alpha$.*

Remark 6.1.4. *In [5], an asymptotic expansion for the trace of relativisitic α-stable processes in bounded $C^{1,1}$ open sets was established. Compared with Theorem 6.1.1, the expansion of [5] does not contain the intermediate terms.*

The rest of the chapter is organized as follows. In Section 6.2, we recall some basic facts about relativistic stable processes and present several preliminary results which will be used in Sections 6.3 and 6.4. Theorem 6.1.1 is proved in Section 6.3, while Theorem 6.1.2 is proved in Section 6.4.

Throughout this chapter, we will use c to denote a positive constant depending (unless otherwise explicitly stated) only on d and α but whose value may change from line to line, even within a single line. In this chapter, the big O notation $f(t) = O(g(t))$ always means that there exist constants C and $t_0 > 0$ such that $f(t) \leq Cg(t)$ for all $0 < t < t_0$.

6.2 Preliminaries

In this section, we recall some basic facts about relativistic α-stable processes. From (6.1.1), one can easily see that X^m has the following approximate scaling property:

$$\{m^{-1/\alpha}(X^1_{mt} - X^1_0), t \geq 0\} \text{ has the same law as } \{X^m_t - X^m_0, t \geq 0\}.$$

In terms of transition densities, this approximate scaling property can be written as

$$p^m(t,x,y) = m^{d/\alpha} p^1(mt, m^{1/\alpha}x, m^{1/\alpha}y). \tag{6.2.1}$$

It is well known that the transition density $p^m_D(t,x,y)$ of $X^{m,D}$ is continuous on $(0,\infty)\times D\times D$. Since both $p^m(t,x,y)$ and $p^m_D(t,x,y)$ are continuous on $(0,\infty)\times D\times D$, $r^m_D(t,x,y)=p^m(t,x,y)-p^m_D(t,x,y)$ is also continuous there. $p^m_D(t,x,y)$ and $r^m_D(t,x,y)$ also enjoy the following approximate scaling property:

$$p^1_{m^{1/\alpha}D}(t,x,y)=m^{-d/\alpha}p^m_D(t/m,x/m^{1/\alpha},y/m^{1/\alpha}), \tag{6.2.2}$$

$$r^1_{m^{1/\alpha}D}(t,x,y)=m^{-d/\alpha}r^m_D(t/m,x/m^{1/\alpha},y/m^{1/\alpha}). \tag{6.2.3}$$

The Lévy measure of the relativistic α-stable process X^m has a density

$$J^m(x)=j^m(|x|):=\frac{\alpha}{2\Gamma(1-\alpha/2)}\int_0^\infty (4\pi u)^{-d/2}e^{-|x|^2/4u}e^{-m^{2/\alpha}u}u^{-(1+\alpha/2)}du,$$

which is continuous and radially decreasing on $\mathbb{R}^d\setminus\{0\}$ (see [49, Lemma 2]). Put $J^m(x,y):=j^m(|x-y|)$. Let $\mathcal{A}(d,-\alpha):=\alpha 2^{\alpha-1}\pi^{-d/2}\Gamma(\frac{d+\alpha}{2})\Gamma(1-\frac{\alpha}{2})^{-1}$. Using change of variables twice, first with $u=|x|^2v$ then with $v=1/s$, we get

$$J^m(x,y)=\mathcal{A}(d,-\alpha)|x-y|^{-d-\alpha}\psi(m^{1/\alpha}|x-y|), \tag{6.2.4}$$

where

$$\psi(r):=2^{-(d+\alpha)}\Gamma\left(\frac{d+\alpha}{2}\right)^{-1}\int_0^\infty s^{(d+\alpha)/2-1}e^{-s/4-r^2/s}ds, \tag{6.2.5}$$

which satisfies $\psi(0)=1$ and

$$c_1^{-1}e^{-r}r^{(d+\alpha-1)/2}\le\psi(r)\le c_1e^{-r}r^{(d+\alpha-1)/2}\quad\text{on }[1,\infty)$$

for some $c_1>1$ (see [26, pp. 276-277] for details). We denote the Lévy density of X by

$$J(x,y):=J^0(x,y)=\mathcal{A}(d,-\alpha)|x-y|^{-d-\alpha}.$$

Note that from (6.2.4) and (6.2.5) we see that for any $x\in\mathbb{R}^d\setminus\{0\}$

$$j^m(|x|)\le j^0(|x|).$$

It follows from [20, Theorem 4.1.] that, for any positive constants M and T there exists a constant $c > 1$ such that for all $m \in (0, M]$, $t \in (0, T]$, and $x, y \in \mathbb{R}^d$ we have

$$c^{-1}\left(t^{-d/\alpha} \wedge tJ^m(x,y)\right) \le p^m(t,x,y) \le c\left(t^{-d/\alpha} \wedge tJ^m(x,y)\right). \tag{6.2.6}$$

We will need a simple lemma from [32] about the relationship between $r^m_D(t,x,y)$ and $r^0_D(t,x,y)$. The lemma is true in much more general situations but we just need it when one of the processes is a symmetric α-stable process and the other is a relativistic α-stable process.

Lemma 6.2.1. *Suppose that X and Y are two Lévy processes with Lévy densities J^X and J^Y, respectively. Suppose that $\sigma = J^X - J^Y$ is nonnegative on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} \sigma(x)dx = \ell < \infty$ and D is an open set. Then for any $x \in D$ and $t > 0$,*

$$p^Y_D(t,x,\cdot) \le e^{\ell t} p^X_D(t,x,\cdot) \quad a.s.$$

If, in addition, $p^X(t,\cdot)$ and $p^Y(t,\cdot)$ are continuous, then we have for $x, y \in D$,

$$r^Y_D(t,x,y) \le e^{2\ell t} r^X_D(t,x,y).$$

The next proposition is the (generalized) Ikeda-Watanabe formula for the relativistic stable process, which describes the joint distribution of τ^m_D and $X^m_{\tau^m_D}$.

Proposition 6.2.2 (Proposition 2.7 [45])**.** *Assume that D is an open subset of $\mathbb{R}^d$ and A is a Borel set such that $A \subset D^c \setminus \partial D$. If $0 \le t_1 < t_2 < \infty$, then*

$$\mathbb{P}_x\left(X^m_{\tau^m_D} \in A,\ t_1 < \tau^m_D < t_2\right) = \int_D \int_{t_1}^{t_2} p^m_D(s,x,y)ds \int_A J^m(y,z)dzdy, \quad x \in D.$$

Now we state a simple lemma about the upper bound of $r^m_D(t,x,y)$, which is an analogue of [3, Lemma 2.1] for stable processes.

Lemma 6.2.3. *Let M, T be positive constants. Then there exists a constant $c = c(d, \alpha, M, T)$ such* *that for all $m \in (0, M]$ and $t \in (0, T]$ we have*

$$r_D^m(t,x,y) \le c\left(t^{-d/\alpha} \wedge \frac{t\psi(m^{1/\alpha}\delta_D(x))}{\delta_D(x)^{d+\alpha}}\right).$$

Proof. Since ψ is eventually decreasing and $\psi(0) = 1 > 0$, there exists a constant $c_1 > 0$ such that $\psi(x) \le c_1\psi(y)$ for all $0 \le y \le x$. Now from the definition of $r_D^m(t,x,y)$ and (6.2.6) we have

$$\begin{aligned} r_D^m(t,x,y) &= r_D^m(t,y,x) \\ &\le \mathbb{E}_y\left[t > \tau_D^m; p^m(t-\tau_D^m, X^m_{\tau_D^m}, x)\right] \\ &\le \mathbb{E}_y\left[c\left(t^{-d/\alpha} \wedge \frac{t\psi(m^{1/\alpha}|x - X^m_{\tau_D^m}|)}{|x - X^m_{\tau_D^m}|^{d+\alpha}}\right)\right] \\ &\le cc_1\left(t^{-d/\alpha} \wedge \frac{t\psi(m^{1/\alpha}\delta_D(x))}{\delta_D(x)^{d+\alpha}}\right). \end{aligned}$$

□

We will need two results from [3]. The first result is about the difference $p_F^m(t,x,y) - p_D^m(t,x,y)$ when $D \subset F$. The proof in [3], given for stable processes, mainly uses the strong Markov property and it works for all strong Markov processes with transition densities.

Proposition 6.2.4 (Proposition 2.3 [3])**.** *Let D and F be open sets in $\mathbb{R}^d$ such that $D \subset F$. Then for any $x, y \in \mathbb{R}^d$ we have*

$$p_F^m(t,x,y) - p_D^m(t,x,y) = \mathbb{E}_x\left[\tau_D^m < t, X^m_{\tau_D^m} \in F \setminus D : p_F^m(t-\tau_D^m, X^m_{\tau_D^m}, y)\right].$$

Now we introduce some notation. Recall that if D is a $C^{1,1}$ open set with characteristics (r_0, Λ_0), then for every $x \in D$ with $\delta_{\partial D}(x) < r_0$ and $y \in \mathbb{R}^d \setminus \bar{D}$ with $\delta_{\partial D}(y) < r_0$, there are $z_x, z_y \in \partial D$ so that $|x - z_x| = \delta_{\partial D}(x)$, $|y - z_y| = \delta_{\partial D}(y)$ and that $B(x_0, r_0) \subset D$ and $B(y_0, r_0) \subset \mathbb{R}^d \setminus \bar{D}$, where $x_0 = z_x + r_0(x - z_x)/|x - z_x|$ and $y_0 = z_y + r_0(y - z_y)/|y - z_y|$. Let $H(x)$ be the half-space containing $B(x_0, r_0)$ such that $\partial H(x)$ contains z_x and is perpendicular to the segment $\overline{z_x z_y}$. The next proposition says that, in case of the symmetric α-stable process, for small t, the quantity $r_D^0(t,x,x)$ can be replaced by $r_{H(x)}^0(t,x,x)$, which was a very crucial step in proving the main result in [3].

Proposition 6.2.5 (Proposition 3.1 of [3]). *Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ open set with characteristics (r_0, Λ_0). Then, for any x with $\delta_{\partial D}(x) < r_0/2$ and $t > 0$ with $t^{1/\alpha} \le r_0/2$, we have*

$$\left| r_D^0(t,x,x) - r_{H(x)}^0(t,x,x) \right| \le \frac{ct^{1/\alpha}}{r_0 t^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_{\partial D}(x)} \right)^{d+\frac{\alpha}{2}-1} \wedge 1 \right).$$

We will need some facts about the "stability" of the surface area of the boundary of $C^{1,1}$ open sets. The following lemma is [7, Lemma 5].

Lemma 6.2.6. *Let D be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristic (r_0, Λ_0) and define for $0 \le q < r_0$,*

$$D_q = \{x \in D : \delta_D(x) > q\}.$$

Then

$$\left(\frac{r_0 - q}{r_0} \right)^{d-1} |\partial D| \le |\partial D_q| \le \left(\frac{r_0}{r_0 - q} \right)^{d-1} |\partial D|, \quad 0 \le q < r_0.$$

The following result is [3, Corollary 2.14].

Lemma 6.2.7. *Let D be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristic (r_0, Λ_0). For any $0 < q \le r_0/2$, we have*

(1) $2^{-d+1}|\partial D| \le |\partial D_q| \le 2^{d-1}|\partial D|$,

(2) $|\partial D| \le \frac{2^d |D|}{r_0}$,

(3) $||\partial D_q| - |\partial D|| \le \frac{2^d dq |\partial D|}{r_0} \le \frac{2^{2d} dq |D|}{r_0^2}$.

6.3 Proof for Bounded $C^{1,1}$ Open Sets

We first prove that $\lim_{t \to 0} t^{\frac{d}{\alpha}} Z_D^m(t)$ exists and identify the limit.

Lemma 6.3.1. *The limit $\lim_{t \to 0} t^{\frac{d}{\alpha}} Z_D^m(t)$ exists and is equal to $C_1|D|$, where C_1 is the constant in Theorem 6.1.1.*

Proof. By definition,

$$\begin{aligned} t^{d/\alpha} Z_D^m(t) &= t^{d/\alpha} \int_D p_D^m(t,x,x)dx \\ &= t^{d/\alpha} \left(\int_D p^m(t,x,x)dx - \int_D r_D^m(t,x,x)dx \right). \end{aligned} \tag{6.3.1}$$

For the first integral on the right hand side of (6.3.1), note that, by the approximate scaling property (6.2.2) and the dominated convergence theorem, we have, as $t \to 0$,

$$\begin{aligned} t^{d/\alpha} \left(\int_D p^m(t,x,x)dx \right) &= \int_D p^{tm}(1,x,x)dx = |D| p^{tm}(1,0) \\ \to \quad |D| \cdot p^0(1,0) &= |D| \cdot \frac{\Gamma(d/\alpha)\omega_d}{(2\pi)^d \alpha}. \end{aligned}$$

It remains to show that $\lim_{t\to 0} t^{d/\alpha} \int_D r_D^m(t,x,x)dx = 0$. By Lemma 6.2.3 we have that

$$t^{d/\alpha} r_D^m(t,x,y) \le c, \qquad (t,x,y) \in (0,1] \times D \times D,$$

for some $c > 0$. Hence we have by the monotone convergence theorem,

$$t^{d/\alpha} \int_{D \setminus D_{t^{1/2\alpha}}} r_D^m(t,x,x) \to 0 \quad \text{ as } t \to 0.$$

For $x \in D_{t^{1/2\alpha}}$ we have by Lemma 6.2.3 again for $t \in (0,1]$,

$$r_D^m(t,x,x) \le c\, t^{\frac{1}{2} + \frac{d}{2\alpha}}, \qquad x \in D_{t^{1/2\alpha}}.$$

Hence $\lim_{t\to 0} t^{d/\alpha} \int_{D_{t^{1/2\alpha}}} r_D^m(t,x,x)dx = 0$. □

It follows from Lemma 6.3.1 that if $N^m(\lambda)$ denotes the number of eigenvalues $\lambda_j^{(m)}$ such that $\lambda_j^m \le \lambda$, then it follows from the classical Karamata Tauberian theorem (see for example [29]) that

$$N^m(\lambda) \sim \frac{C_1 |D|}{\Gamma(d/\alpha + 1)} \lambda^{d/\alpha}, \quad \text{ as } \lambda \to \infty.$$

This is the analogue for killed relativistic stable processes of the celebrated Weyl's asymptotic formula for the eigenvalues of the Dirichlet Laplacian and it is already proved in [5] (see [5, (1.10)]). This result has been known at least since 2009, see [5, Remark 1.2].

Now we focus on identifying the next terms in $Z_D^m(t)$. For this, we need to find the order of t in $Z_D^m(t) - C_1 t^{-\frac{d}{\alpha}}$. Note that by Lemma 6.3.1,

$$\begin{aligned} Z_D^m(t) - C_1 t^{-d/\alpha} &= \int_D p_D^m(t,x,x) - p^0(t,x,x)dx \\ = \quad \int_D \left(p^m(t,x,x) - p^0(t,x,x)\right) dx &- \int_D r_D^m(t,x,x)dx. \end{aligned}$$

The next lemma gives the orders of t in $p^m(t,x,x) - p^0(t,x,x)$ up to $t^{\frac{2}{\alpha}} t^{-\frac{d}{\alpha}}$.

Lemma 6.3.2. *Let k be the largest integer such that $k < \frac{2}{\alpha}$. Then we have*

$$p^m(t,x,x) - p^0(t,x,x) = t^{-d/\alpha} \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha} \sum_{n=1}^{k} \frac{m^n}{n!} t^n + O(t^{2/\alpha} t^{-d/\alpha}).$$

Proof. By the scaling property (6.2.1) we have

$$\begin{aligned} & p^m(t,x,x) - p^0(t,x,x) = p^m(t,0) - p^0(t,0) \\ = \quad & t^{-d/\alpha}\left(p^{tm}(1,0) - p^0(1,0)\right) \\ = \quad & t^{-d/\alpha}(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-(|\xi|^2 + (mt)^{2/\alpha})^{\alpha/2} + mt} - e^{-|\xi|^\alpha} d\xi. \end{aligned}$$

Note that for any $x \geq 0$ we have $(1+x)^{\alpha/2} \leq 1 + \frac{\alpha}{2}x$. Thus

$$\left(|\xi|^2 + (mt)^{2/\alpha}\right)^{\alpha/2} = |\xi|^\alpha \left(1 + \frac{(mt)^{2/\alpha}}{|\xi|^2}\right)^{\alpha/2} \leq |\xi|^\alpha \left(1 + \frac{\alpha}{2} \frac{(mt)^{2/\alpha}}{|\xi|^2}\right).$$

Consequently

$$
\begin{aligned}
&0 \le e^{-|\xi|^\alpha} - e^{-\left(|\xi|^2+(mt)^{2/\alpha}\right)^{\alpha/2}} \\
\le\;& e^{-|\xi|^\alpha} - e^{-|\xi|^\alpha\left(1+\frac{\alpha}{2}\frac{(mt)^{2/\alpha}}{|\xi|^2}\right)} = e^{-|\xi|^\alpha}\left(1-e^{-\frac{\alpha}{2}\frac{(mt)^{2/\alpha}}{|\xi|^{2-\alpha}}}\right) \\
\le\;& e^{-|\xi|^\alpha}\left(\frac{\alpha}{2}\frac{(mt)^{2/\alpha}}{|\xi|^{2-\alpha}}\right),
\end{aligned}
$$

where we used $1-e^{-x} \le x$ for all $x \ge 0$ in the last inequality above. Therefore

$$
\begin{aligned}
&0 \le \int_{\mathbb{R}^d} e^{-\left(|\xi|^2+(mt)^{2/\alpha}\right)^{\alpha/2}+mt} - e^{-|\xi|^\alpha} d\xi \\
\le\;& \int_{\mathbb{R}^d} \left| e^{-\left(|\xi|^2+(mt)^{2/\alpha}\right)^{\alpha/2}+mt} - e^{-|\xi|^\alpha}e^{mt} + e^{-|\xi|^\alpha}e^{mt} - e^{-|\xi|^\alpha} \right| d\xi \\
\le\;& \int_{\mathbb{R}^d} \left| e^{-\left(|\xi|^2+(mt)^{2/\alpha}\right)^{\alpha/2}+mt} - e^{-|\xi|^\alpha}e^{mt} \right| d\xi + \int_{\mathbb{R}^d} \left| e^{-|\xi|^\alpha}e^{mt} - e^{-|\xi|^\alpha} \right| d\xi \\
\le\;& \int_{\mathbb{R}^d} e^{mt}e^{-|\xi|^\alpha}\left(\frac{\alpha}{2}\frac{(mt)^{2/\alpha}}{|\xi|^{2-\alpha}}\right) d\xi + \int_{\mathbb{R}^d} e^{-|\xi|^\alpha}\left(e^{mt}-1\right) d\xi \\
=\;& e^{mt}\frac{\alpha}{2}(mt)^{2/\alpha}\int_{\mathbb{R}^d} \frac{e^{-|\xi|^\alpha}}{|\xi|^{2-\alpha}} d\xi + \sum_{n=1}^{\infty}\frac{(mt)^n}{n!}\int_{\mathbb{R}^d} e^{-|\xi|^\alpha} d\xi.
\end{aligned}
$$

Since $k+j \ge 2/\alpha$ for any $j \ge 1$, we have $\displaystyle\sum_{n=k+1}^{\infty}\frac{(mt)^n}{n!} = O(t^{2/\alpha})$. Therefore

$$
\int_{\mathbb{R}^d}\left(e^{-\left(|\xi|^2+(mt)^{2/\alpha}\right)^{\alpha/2}+mt} - e^{-|\xi|^\alpha}\right) d\xi = O(t^{2/\alpha}) + \frac{\omega_d\Gamma(d/\alpha)}{\alpha}\sum_{n=1}^{k}\frac{(mt)^n}{n!}
$$

and

$$
p^m(t,x,x) - p^0(t,x,x) = t^{-d/\alpha}\frac{\omega_d\Gamma(d/\alpha)}{(2\pi)^d\alpha}\sum_{n=1}^{k}\frac{m^n}{n!}t^n + O(t^{2/\alpha}t^{-d/\alpha}).
$$

□

Now we try to find the orders of t in the expansion of $\int_D r_D^m(t,x,x)dx$ up to the order of $t^{\frac{2}{\alpha}}t^{-\frac{d}{\alpha}}$. For this, we need to assume some regularity condition on the boundary of D. Hence in the remainder of this section we assume that D is a bounded $C^{1,1}$ open set with characteristic (r_0, Λ_0). We also assume that $t^{1/\alpha} \le \frac{r_0}{2}$.

We first deal with the contribution in $D_{r_0/2}$.

Lemma 6.3.3. *There exists $c = c(d, \alpha) > 0$ such that*

$$\int_{D_{r_0/2}} r_D^m(t, x, x)dx \le ce^{2mt} \frac{|D|t^{2/\alpha}}{r_0^2 t^{d/\alpha}}.$$

Proof. It follows from Lemma 6.2.1 that $r_D^m(t, x, y) \le e^{2mt} r_D^0(t, x, y)$. By [3, (3.2)] we know that

$$\int_{D_{r_0/2}} r_D^0(t, x, y)dx \le \frac{c|D|t^{2/\alpha}}{r_0^2 t^{d/\alpha}}.$$

The desired assertion follows immediately. □

Lemma 6.3.4. *There exists $c = c(d, \alpha) > 0$ such that*

$$r_D^m(t, x, x) - r_{H(x)}^m(t, x, x) \le ce^{2mt} \frac{t^{1/\alpha}}{t^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\frac{\alpha}{2}-1} \wedge 1 \right)$$

and

$$\int_{D \setminus D_{r_0/2}} \left(r_D^m(t, x, x) - r_{H(x)}^m(t, x, x) \right) dx \le ce^{2mt} \frac{t^{2/\alpha}}{t^{d/\alpha}}.$$

Proof. If the first assertion of the lemma is right, then it is easy to see that

$$\int_{D \setminus D_{r_0/2}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\frac{\alpha}{2}-1} \wedge 1 \right) dx \le ct^{1/\alpha}.$$

Hence we focus on proving the first assertion. By [3, (3.4)], we know that

$$r_D^0(t, x, x) - r_{H(x)}^0(t, x, x) \le c \frac{t^{1/\alpha}}{t^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\frac{\alpha}{2}-1} \wedge 1 \right).$$

Recall that $J^m(x) \le J^0(x)$ for any $x \in \mathbb{R}^d \setminus \{0\}$. Now it follows from the generalized Ikeda-Watanabe formula and Lemma 6.2.1 that

$$\begin{aligned}
& r_D^m(t,x,x) - r_{H(x)}^m(t,x,x) \\
= \;& \mathbb{E}_x \left[t > \tau_D^m, X^m_{\tau_D^m} \in H(x) \setminus D; p^m_{H(x)}(t-\tau_D^m, X^m_{\tau_D^m}, x) \right] \\
= \;& \int_D \int_0^t p_D^m(s,x,y)ds \int_{H(x)\setminus D} J^m(y,z) p^m_{H(x)}(t-s,z,x)dzdy \\
\leq \;& e^{2mt} \int_D \int_0^t p_D^0(s,x,y)ds \int_{H(x)\setminus D} J^0(y,z) p^0_{H(x)}(t-s,z,x)dzdy \\
= \;& e^{2mt} \mathbb{E}_x \left[t > \tau_D^0, X_{\tau_D^0} \in H(x) \setminus D; p^0_{H(x)}(t-\tau_D^0, X_{\tau_D^0}, x) \right] \\
= \;& e^{2mt} \left(r_D^0(t,x,x) - r^0_{H(x)}(t,x,x) \right) \\
\leq \;& c e^{2mt} \frac{t^{1/\alpha}}{t^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)}\right)^{d+\frac{\alpha}{2}-1} \wedge 1 \right).
\end{aligned}$$

□

Lemma 6.3.5. *There exists $c = c(d,\alpha) > 0$ such that*

$$\int_{D\setminus D_{r_0/2}} r^m_{H(x)}(t,x,x)dx - t^{1/\alpha} t^{-d/\alpha} \int_0^{\frac{r_0}{2t^{1/\alpha}}} |\partial D| f_H^{tm}(1,u)du \leq c t^{2/\alpha} t^{-d/\alpha}.$$

Proof. Using the scaling relation (6.2.3) we get

$$\begin{aligned}
& \int_{D\setminus D_{r_0/2}} r^m_{H(x)}(t,x,x)dx \\
= \;& \int_0^{r_0/2} |\partial D_u| f_H^m(t,u)du \\
= \;& \int_0^{r_0/2} |\partial D_u| t^{-d/\alpha} f_H^{tm}(1, u/t^{1/\alpha})du \\
= \;& t^{1/\alpha} t^{-d/\alpha} \int_0^{r_0/2t^{1/\alpha}} |\partial D_{ut^{1/\alpha}}| f_H^{tm}(1,u)du.
\end{aligned}$$

It follows from Corollary 6.2.7 that $||\partial D_q| - |\partial D|| \leq \frac{2^d dq|\partial D|}{r_0} \leq \frac{2^{2d} dq|D|}{r_0^2}$ for any $q \leq r_0/2$. Hence

$$\begin{aligned} & \left| \int_{D \setminus D_{r_0/2}} r^m_{H(x)}(t,x,x) - t^{1/\alpha} t^{-d/\alpha} \int_0^{\frac{r_0}{2t^{1/\alpha}}} |\partial D| f^{tm}_H(1,u) du \right| \\ \leq\ & t^{1/\alpha} t^{-d/\alpha} \int_0^{\frac{r_0}{2t^{1/\alpha}}} ||\partial D_{ut^{1/\alpha}}| - |\partial D||\, f^{tm}_H(1,u) du \\ \leq\ & c_1 t^{2/\alpha} t^{-d/\alpha} \int_0^{\infty} u f^{tm}_H(1,u) du \\ \leq\ & c_2 t^{2/\alpha} t^{-d/\alpha}. \end{aligned}$$

□

Lemma 6.3.6. *There exists $c = c(d, \alpha) > 0$ such that*

$$t^{1/\alpha} t^{-d/\alpha} \int_0^{\infty} |\partial D| f^{tm}_H(1,u) du - t^{1/\alpha} t^{-d/\alpha} \int_0^{\frac{r_0}{2t^{1/\alpha}}} |\partial D| f^{tm}_H(1,u) du \leq c t^{2/\alpha} t^{-d/\alpha}.$$

Proof. It follows from Lemma 6.2.1 that

$$\begin{aligned} & t^{1/\alpha} t^{-d/\alpha} \int_0^{\infty} |\partial D| f^{tm}_H(1,u) du - t^{1/\alpha} t^{-d/\alpha} \int_0^{\frac{r_0}{2t^{1/\alpha}}} |\partial D| f^{tm}_H(1,u) du \\ =\ & t^{1/\alpha} t^{-d/\alpha} \int_{\frac{r_0}{2t^{1/\alpha}}}^{\infty} |\partial D| f^{tm}_H(1,u) du \\ =\ & t^{1/\alpha} t^{-d/\alpha} |\partial D| \int_{\frac{r_0}{2t^{1/\alpha}}}^{\infty} f^{tm}_H(1,u) du \\ \leq\ & e^{2mt} t^{1/\alpha} t^{-d/\alpha} |\partial D| \int_{\frac{r_0}{2t^{1/\alpha}}}^{\infty} f^{0}_H(1,u) du. \end{aligned}$$

For $q \geq r_0/(2t^{1/\alpha})$ we have $f^0_H(1,q) \leq cq^{-d-\alpha} \leq cq^{-2}$. Hence

$$\int_{\frac{r_0}{2t^{1/\alpha}}}^{\infty} f^{0}_H(1,u) du \leq c \int_{\frac{r_0}{2t^{1/\alpha}}}^{\infty} \frac{dq}{q^2} \leq c \frac{t^{1/\alpha}}{r_0}$$

and the result now follows. □

Lemma 6.3.7. $\lim_{t \downarrow 0} \int_0^{\infty} f^{tm}_H(1,u) du = \int_0^{\infty} f^{0}_H(1,u) du.$

Proof. This follows immediately from the continuity of $m \mapsto r_D^m(t,x,y)$ and the dominated convergence theorem. □

Proof of Theorem 6.1.1 Combining Lemmas 6.3.1, 6.3.2, 6.3.3, 6.3.4, 6.3.5, 6.3.6, and 6.3.7, we immediately arrive at Theorem 6.1.1. □

6.4 Proof for Bounded Lipschitz Open Sets

In this section we always assume that D is a bounded Lipschitz open set in $\mathbb{R}^d$. The argument of this section is similar to previous section and [4]. We will follow the argument in [4] closely, making necessary modifications for relativistic stable processes. Note that even though the main theorem in [4] is stated for a Lipschitz domain, it remains true for a bounded Lipschitz open set.

First we need two technical facts which play crucial roles later. The first proposition is [4, Proposition 2.9] and we will state it here for reader's convenience.

Proposition 6.4.1 (Proposition 2.9. [4])**.** *Suppose that $f : (0,\infty) \to [0,\infty)$ is continuous and satisfies $f(r) \le c(1 \wedge r^{-\beta})$ for some $\beta > 1$. Furthermore, suppose that for any $0 < R_1 < R_2 < \infty$, f is Lipschitz on $[R_1, R_2]$. Then we have*

$$\lim_{\eta \to 0^+} \frac{1}{\eta} \int_D f\left(\frac{\delta_D(x)}{\eta}\right) dx = \mathcal{H}^{d-1}(\partial D) \int_0^\infty f(r)dr.$$

Lemma 6.4.2. *Suppose that $f : (0,\infty) \to [0,\infty)$ is continuous and satisfies $f(r) \le c_1(1 \wedge r^{-\beta})$ for some $\beta > 1$. Furthermore, suppose that for any $0 < R_1 < R_2 < \infty$, f is Lipschitz on $[R_1, R_2]$. Let $\{f^\eta : \eta > 0\}$ be continuous functions from $(0,\infty)$ to $[0,\infty)$ such that, for any $0 < L < M < \infty$, $\lim_{\eta \to 0} f^\eta(r) = f(r)$ uniformly for $r \in [L, M]$. Suppose that there exists $c_2 > 0$ such that $f^\eta(r) \le c_2 f(r)$ for all $\eta \le 1$. Then we have*

$$\lim_{\eta \to 0^+} \frac{1}{\eta} \int_D f^\eta\left(\frac{\delta_D(x)}{\eta}\right) dx = \mathcal{H}^{d-1}(\partial D) \int_0^\infty f(r)dr.$$

Proof. Let $\psi_\eta(r) = \eta^{-1} \left|\{x \in D : \delta_D(x) < \eta r\}\right|$. Note (cf. proof of [14, Proposition 1.1]) that

$\psi_\eta(r) \le c$ for all $\eta, r > 0$ and that

$$\eta^{-1} \int_D f\left(\frac{\delta_D(x)}{\eta}\right) dx = \int_0^\infty f(r) d\psi_\eta(r),$$

and

$$\eta^{-1} \int_D f^\eta \left(\frac{\delta_D(x)}{\eta}\right) dx = \int_0^\infty f^\eta(r) d\psi_\eta(r).$$

It was shown in [4, Proposition 2.9.] that, for any $0 < R_1 < R_2 < \infty$ and $\eta > 0$, f satisfies

$$\int_0^{R_1} f(r) d\psi_\eta(r) \le cR_1, \tag{6.4.1}$$

$$\int_{R_2}^\infty f(r) d\psi_\eta(r) \le c\eta^{\beta-1} + cR_2^{1-\beta}, \tag{6.4.2}$$

$$\lim_{\eta \to 0^+} \int_{R_1}^{R_2} f(r) d\psi_\eta(r) = \mathcal{H}^{d-1}(\partial D) \int_{R_1}^{R_2} f(r) dr.$$

Since $f^\eta \le c_2 f$ for $\eta \le 1$ we have the same inequalities as (6.4.1) and (6.4.2) for f^η, $\eta \le 1$. Hence it is enough to show that

$$\lim_{\eta \to 0^+} \int_{R_1}^{R_2} f^\eta(r) d\psi_\eta(r) = \mathcal{H}^{d-1}(\partial D) \int_{R_1}^{R_2} f(r) dr.$$

For any partition $R_1 = x_0 < x_1 < \cdots < x_n = R_2$ of $[R_1, R_2]$, we have

$$\begin{aligned} & \left| \sum_{i=1}^n f^\eta(x_i) \left(\psi_n(x_i) - \psi_n(x_{i-1})\right) - \sum_{i=1}^n f(x_i) \left(\psi_n(x_i) - \psi_n(x_{i-1})\right) \right| \\ = \quad & \sum_{i=1}^n |f^\eta(x_i) - f(x_i)| \left(\psi_n(x_i) - \psi_n(x_{i-1})\right) \\ \le \quad & \| f^\eta - f \|_{L^\infty[R_1, R_2]} \psi_\eta(R_2). \end{aligned}$$

Note that for any $\eta > 0$ the function $r \to \psi_\eta(r)$ is nondecreasing and for any $\eta > 0$, $r > 0$ we have $\psi_\eta(r) \le cr$ for some constant c. Since $f^\eta \to f$ uniformly on $r \in [R_1, R_2]$, taking supremum for all possible partitions gives

$$\lim_{\eta \to 0^+} \int_{R_1}^{R_2} f^\eta(r) d\psi_\eta(r) = \lim_{\eta \to 0^+} \int_{R_1}^{R_2} f(r) d\psi_\eta(r) = \mathcal{H}^{d-1}(\partial D) \int_{R_1}^{R_2} f(r) dr.$$

□

Lemma 6.4.3. *For any* $0 < L < M < \infty$, $p^m(t,x,y)$ *converges uniformly to* $p^0(t,x,y)$ *as* $m \to 0$ *for* $(t,x,y) \in [L,M] \times \mathbb{R}^d \times \mathbb{R}^d$.

Proof. Note that

$$\begin{aligned}
&\left|p^m(t,x,y) - p^0(t,x,y)\right| = \left|(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi(y-x)} \left(e^{-t((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m)} - e^{-t|\xi|^\alpha}\right) d\xi\right| \\
\leq\ & (2\pi)^{-d} \int_{\mathbb{R}^d} \left|e^{-i\xi(y-x)} \left(e^{-t((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m)} - e^{-t|\xi|^\alpha}\right)\right| d\xi \\
\leq\ & (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m)} - e^{-t|\xi|^\alpha} d\xi \\
=\ & (2\pi)^{-d}(p^m(t,0) - p^0(t,0)).
\end{aligned}$$

Now it follows from Lemma 6.3.2 that for $t \in [L,M]$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
&\left|p^0(t,x,y) - p^m(t,x,y)\right| \\
\leq\ & t^{-d/\alpha}(2\pi)^{-d} e^{mt} \frac{\alpha}{2}(mt)^{2/\alpha} \int_{\mathbb{R}^d} \frac{e^{-|\xi|^\alpha}}{|\xi|^{2-\alpha}} d\xi + \sum_{n=1}^{\infty} \frac{(mt)^n}{n!} \int_{\mathbb{R}^d} e^{-|\xi|^\alpha} d\xi \\
\leq\ & L^{-d/\alpha}(2\pi)^{-d} \frac{\alpha}{2}(mM)^{2/\alpha} \int_{\mathbb{R}^d} \frac{e^{-|\xi|^\alpha}}{|\xi|^{2-\alpha}} d\xi + \sum_{n=1}^{\infty} \frac{(mM)^n}{n!} \int_{\mathbb{R}^d} e^{-|\xi|^\alpha} d\xi.
\end{aligned}$$

The last quantity above converges to 0 as $m \to 0$. □

For convenience, we define the following notation.

$$f_H^m(t,r) := r_H^m(t, (r,\tilde{0}), (r,\tilde{0})), \qquad r > 0.$$

Lemma 6.4.4. *For any* $0 < L < M < \infty$ *and* $m > 0$,

$$\lim_{t \to 0} f_H^{tm}(1,r) = f_H^0(1,r), \quad \textit{uniformly in } r \in [L,M],$$

that is, given $\varepsilon > 0$ *there exists* $t_0 > 0$ *such that for* $0 \leq t \leq t_0$ *we have*

$$\sup_{r \in [L,M]} \left|r_H^{tm}(1,(r,\tilde{0}),(r,\tilde{0})) - r_H^0(1,(r,\tilde{0}),(r,\tilde{0}))\right| < \varepsilon.$$

Proof. Recall that $r_H^0(t,x,y) = \mathbb{E}_x[\tau_H^0 < t, p^0(t-\tau_H^0, X_{\tau_H^0}, y)]$ and $r_H^m(t,x,y) = \mathbb{E}_x[\tau_H^m < t, p^m(t-\tau_H^m, X^m_{\tau_H^m}, y)]$. It is well known that

$$p^0(t,x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}.$$

Since $|X^0_{\tau_H} - (r,\tilde{0})| > L$, we have, together with Lemma 6.2.1,

$$p^0(1-\tau_H^0, X^0_{\tau_H^0}, (r,\tilde{0})) \leq c\frac{1-\tau_H^0}{L^{d+\alpha}},$$

$$p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0})) \leq ce^{tm}\frac{1-\tau_H^{tm}}{L^{d+\alpha}}.$$

Now take δ_1 small so that

$$\mathbb{E}_{(r,\tilde{0})}\left[1-\delta_1 \leq \tau_H^0 < 1, p^0(1-\tau_H^0, X^0_{\tau_H^0}, (r,\tilde{0}))\right] < \varepsilon, \tag{6.4.3}$$

$$\mathbb{E}_{(r,\tilde{0})}\left[1-\delta_1 \leq \tau_H^{tm} < 1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))\right] < \varepsilon. \tag{6.4.4}$$

Now let V^m be a Lévy process with Lévy density $\sigma = J - J^m$ and define $T^m := \inf\{t > 0 : V_t^m \neq 0\}$. Then V^m is a compound Poisson process and T^m is an exponential random variable with parameter m and independent of X (See [49]). Then we have

$$\begin{aligned}
& \mathbb{E}_{(r,\tilde{0})}\left[\tau_H^{tm} < 1-\delta_1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))\right] \\
= \;& \mathbb{E}_{(r,\tilde{0})}\left[T^{tm} > 1, \tau_H^{tm} < 1-\delta_1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))\right] \\
+ \;& \mathbb{E}_{(r,\tilde{0})}\left[T^{tm} \leq 1, \tau_H^{tm} < 1-\delta_1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))\right].
\end{aligned}$$

Since $p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0})) \leq c\frac{e^{mt}}{L^{d+\alpha}}$, we have

$$\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} \leq 1, \tau_H^{tm} < 1-\delta_1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))\right] \leq c\frac{e^{mt}}{L^{d+\alpha}}(1-e^{-mt}). \tag{6.4.5}$$

Similarly we also have

$$\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} \leq 1, \tau_H^0 < 1-\delta_1, p^0(1-\tau_H^0, X^0_{\tau_H}, (r,\tilde{0}))\right] \leq c\frac{1}{L^{d+\alpha}}(1-e^{-mt}). \tag{6.4.6}$$

Take $t_1 > 0$ such that (6.4.5) and (6.4.6) is less than ε for all $t \le t_1$. Next note that for $T^{tm} > 1$ and $\tau_H^{tm} < 1$, we have $\tau_H^{tm} = \tau_H^0$ and $X^{tm}_{\tau_H^{tm}} = X^0_{\tau_H^0}$. Hence it follows that

$$\begin{aligned}
& |\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} > 1, \tau_H^{tm} < 1-\delta_1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))\right] \\
& -\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} > 1, \tau_H^{0} < 1-\delta_1, p^{0}(1-\tau_H^{0}, X^{0}_{\tau_H^{0}}, (r,\tilde{0}))\right]| \\
\le\ & \mathbb{E}_{(r,\tilde{0})}\left[T^{tm} > 1, \tau_H^{0} < 1-\delta_1, |p^{tm}(1-\tau_H^{0}, X^{0}_{\tau_H^{0}}, (r,\tilde{0})) - p^{0}(1-\tau_H^{0}, X^{0}_{\tau_H^{0}}, (r,\tilde{0}))|\right] \\
\le\ & \sup_{s\in[\delta_1,1],x,y\in\mathbb{R}^d} |p^{tm}(s,x,y) - p^0(s,x,y)|. \qquad (6.4.7)
\end{aligned}$$

It follows from Lemma 6.4.3 that there exists $t_2 > 0$ such that $\sup_{s\in[\delta_1,1],x,y\in\mathbb{R}^d} |p^{tm}(s,x,y) - p^0(s,x,y)| < \varepsilon$ for $0 \le t \le t_2$. Now let $t_0 = t_1 \wedge t_2$. Then for any $0 \le t \le t_0$ we have from (6.4.3), (6.4.4), (6.4.5), (6.4.6), and (6.4.7)

$$\begin{aligned}
& |r_H^{tm}(1,(r,\tilde{0}),(r,\tilde{0})) - r_H^0(1,(r,\tilde{0}),(r,\tilde{0}))| \\
=\ & |\mathbb{E}_{(r,\tilde{0})}[\tau_H^{tm} < 1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))] - \mathbb{E}_{(r,\tilde{0})}[\tau_H^0 < 1, p^0(1-\tau_H^0, X^0_{\tau_H^0}, (r,\tilde{0}))]| \\
\le\ & |\mathbb{E}_{(r,\tilde{0})}[1 > \tau_H^{tm} > 1-\delta_1, \tau_H^{tm} < 1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))]| + \\
& |\mathbb{E}_{(r,\tilde{0})}[1 > \tau_H^{0} > 1-\delta_1, \tau_H^{0} < 1, p^{0}(1-\tau_H^{0}, X^{0}_{\tau_H^{0}}, (r,\tilde{0}))]| + \\
& |\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} \le 1, \tau_H^{tm} < 1-\delta_1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))\right]| + \\
& |\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} \le 1, \tau_H^{0} < 1-\delta_1, p^{0}(1-\tau_H^{0}, X^{0}_{\tau_H^{0}}, (r,\tilde{0}))\right]| \\
& +|\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} > 1, \tau_H^{0} < 1-\delta_1, p^{tm}(1-\tau_H^{0}, X^{0}_{\tau_H^{0}}, (r,\tilde{0}))\right] \\
& -\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} > 1, \tau_H^{0} < 1-\delta_1, p^{0}(1-\tau_H^{0}, X^{0}_{\tau_H^{0}}, (r,\tilde{0}))\right]| \\
<\ & 5\varepsilon.
\end{aligned}$$

□

As in [4], we need to divide the Lipschitz open set D into a good set and a bad set. We recall several geometric facts about the Lipschitz open set.

Definition 6.4.5. *Let $\varepsilon, r > 0$. We say that $G \subset \partial D$ is (ε, r)-good if for each point $p \in G$, the* *unit inner normal* $\nu(p)$ *exists and*

$$B(p,r) \cap \partial D \subset \{x : |(x-p)\cdot\nu(p)| < \varepsilon|x-p|\}.$$

If G is an (ε, r)-good subset of ∂D, then using this definition we can construct a good subset $\mathcal{G}$ of the points near the boundary:

$$\mathcal{G} = \bigcup_{p\in G} \Gamma_r(p,\varepsilon),$$

where $\Gamma_r(p,\varepsilon) = \{x : (x-p)\cdot\nu(p) > \sqrt{1-\varepsilon^2}|x-p|\} \cap B(p,r)$.

The next lemma is [4, Lemma 2.7] and it says the measure of the set of the bad points near the boundary is small. Note that even though [4, Lemma 2.7] is stated for a bounded Lipschitz domain, the proof remains true for a bounded Lipschitz open set.

Lemma 6.4.6 (Lemma 2.7 in [4])**.** *Suppose* $\varepsilon \in (0, 1/2)$*,* $r > 0$ *and that* G *is a measurable* (ε, r)*-good subset of* ∂D*. There exists* $s_0(\partial D, G) > 0$ *such that for all* $s < s_0$

$$|\{x \in D : \delta_D(x) < s\} \setminus \mathcal{G}| \le s\left[\mathcal{H}^{d-1}(\partial D \setminus G) + \varepsilon\left(3 + \mathcal{H}^{d-1}(\partial D)\right)\right].$$

The next lemma is about the existence of a good subset $G \subset \partial D$. Again the lemma remains true for a bounded Lipschitz open set D.

Lemma 6.4.7 (Lemma 2.8 in [4])**.** *For any* $\varepsilon > 0$ *there exists* $r > 0$ *such that an* (ε, r)*-good set* $G \subset \partial D$ *exists and*

$$\mathcal{H}^{d-1}(\partial D \setminus G) < \varepsilon.$$

The two lemmas above imply that

$$|\{x \in D : \delta_D(x) < s\} \setminus \mathcal{G}| \le s\varepsilon\left(4 + \mathcal{H}^{d-1}(\partial D)\right).$$

For any $\varepsilon \in (0, 1/4)$, we fix the (ε, r)-good set from Lemma 6.4.7 and construct $\mathcal{G}$ from G. We choose r to be smaller than the minimal distances between (finitely many) components of D. For any $x \in \mathcal{G}$, there exists $p(x) \in \partial D$ such that $x \in \Gamma_r(p(x), \varepsilon)$. Next we define inner and outer cones as follows

$$I_r\left(p(x)\right)=\{y:(y-p(x))\cdot\nu\left(p(x)\right)>\varepsilon|y-p(x)|\}\cap B(p(x),r), \tag{6.4.8}$$

$$U_r\left(p(x)\right)=\{y:(y-p(x))\cdot\nu\left(p(x)\right)<-\varepsilon|y-p(x)|\}\cap B(p(x),r). \tag{6.4.9}$$

It follows from [4, (2.20)] that there exists a half-space $H^*(x)$ such that

$$x\in H^*(x),\quad \delta_{H^*(x)}(x)=\delta_D(x),\quad I_r\left(p(x)\right)\subset H^*(x)\subset U_r\left(p(x)\right)^c. \tag{6.4.10}$$

Now we are ready to prove Theorem 6.1.2.

Proof of Theorem 6.1.2. Fix $\varepsilon\in(0,1/4)$, the (ε,r)-good set from Lemma 6.4.7 and the $\mathcal{G}$ constructed from G. From the definition of the trace we have

$$\begin{aligned}
&-t^{d/\alpha}\int_D r_D^m(t,x,x)dx=t^{d/\alpha}\int_D\left(p_D^m(t,x,x)-p^m(t,x,x)\right)dx\\
=&\ \ t^{d/\alpha}Z_D^m(t)-t^{d/\alpha}\int_D p^m(t,x,x)dx\\
=&\ \ t^{d/\alpha}Z_D^m(t)-t^{d/\alpha}\int_D\left(p^0(t,x,x)-\left(p^0(t,x,x)-p^m(t,x,x)\right)\right)dx\\
=&\ \ t^{d/\alpha}Z_D^m(t)-C_1|D|+t^{d/\alpha}\int_D\left(p^0(t,x,x)-p^m(t,x,x)\right)dx.
\end{aligned}$$

Hence it follows from Lemma 6.3.2 that in order to prove Theorem 6.1.2 we must show that for given $\varepsilon\in(0,1/4)$ there exists a $t_0>0$ such that for any $0<t<t_0$,

$$\left|t^{d/\alpha}\int_D r_D^m(t,x,x)dx-C_2\mathcal{H}^{d-1}(\partial D)t^{1/\alpha}\right|\le c(\varepsilon)t^{1/\alpha},$$

where $c(\varepsilon)\to 0$ as $\varepsilon\to 0$. As in the proof of [4, Theorem 1.1.] we split the region of integration into three sets

$$\mathcal{D}_1=\{x\in D\setminus\mathcal{G}:\delta_D(x)<s\},$$

$$\mathcal{D}_2=\{x\in D\cap\mathcal{G}:\delta_D(x)<s\},$$

$$\mathcal{D}_3=\{x\in D:\delta_D(x)\ge s\},$$

where s must be smaller than the s_0 given by Lemma 6.4.6. For small enough t we can take

$$s = t^{1/\alpha}/\sqrt{\varepsilon}.$$

It is shown in [4, (3.2) and (3.4)] that

$$t^{d/\alpha}\int_{\mathcal{D}_1\cup\mathcal{D}_3} r_D^0(t,x,x)dx \le c(\varepsilon)t^{1/\alpha} \tag{6.4.11}$$

where $c(\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence it follows from Lemma 6.2.1 and (6.4.11) that

$$t^{d/\alpha}\int_{\mathcal{D}_1\cup\mathcal{D}_3} r_D^m(t,x,x)dx \le c(\varepsilon)e^{2mt}t^{1/\alpha}. \tag{6.4.12}$$

Now we deal with the integral on $\mathcal{D}_2$. Let $H^*(x)$, $I_r\left(p(x)\right)$, $U_r\left(p(x)\right)$ be defined by (6.4.8), (6.4.9) and (6.4.10). We have

$$I_r\left(p(x)\right)\subset H^*(x)\subset U_r\left(p(x)\right)^c.$$

Since r is less than the minimal distances between components of D, we also have

$$I_r\left(p(x)\right)\subset D\subset U_r\left(p(x)\right)^c.$$

Since $I_r\left(p(x)\right)\subset U_r\left(p(x)\right)^c$, By an argument similar to that used in Lemma 6.3.4 we have

$$\begin{aligned}
&\left|r_D^m(t,x,x)-r_{H^*(x)}^m(t,x,x)\right|\\
\le\;& r_{I_r(p(x))}^m(t,x,x)-r_{U_r(p(x))^c}^m(t,x,x)\\
\le\;& e^{2mt}\left(r_{I_r(p(x))}^0(t,x,x)-r_{U_r(p(x))^c}^0(t,x,x)\right).
\end{aligned} \tag{6.4.13}$$

Now it follows from [4, Proposition 3.1.] and (6.4.13) that

$$\begin{aligned}
&t^{d/\alpha}\int_{\mathcal{D}_2}\left|r_D^m(t,x,x)-r_{H^*(x)}^m(t,x,x)\right|dx\\
\le\;& ce^{2mt}\left(\varepsilon^{1-\alpha/2}\vee\sqrt{\varepsilon}\right)\mathcal{H}^{d-1}(\partial D)t^{1/\alpha}\int_0^\infty\left(r^{-d-\alpha+1}\wedge 1\right)dr.
\end{aligned}$$

Finally we will show that the integral

$$t^{d/\alpha} \int_{\mathcal{D}_2} r^m_{H^*(x)}(t,x,x)dx$$

gives the second term $C_2 \mathcal{H}^{d-1}(\partial D) t^{1/\alpha}$ plus an error term of order $c(\varepsilon)t^{1/\alpha}$. Recall that

$$r^m_{H^*(x)}(t,x,x) = f^m_{H^*}(t, \delta_{H^*(x)}) = f^m_H(t, \delta_D(x)).$$

Hence we have

$$\begin{aligned} & t^{d/\alpha} \int_{\mathcal{D}_2} r^m_{H^*(x)}(t,x,x)dx \\ = \;& t^{d/\alpha} \int_{\mathcal{D}_2} f^m_H(t, \delta_D(x))dx \\ = \;& t^{d/\alpha} \int_D f^m_H(t, \delta_D(x))dx - t^{d/\alpha} \int_{\mathcal{D}_1 \cup \mathcal{D}_3} f^m_H(t, \delta_D(x))dx. \end{aligned}$$

By an argument similar to that used to get (6.4.12) we have that

$$t^{d/\alpha} \int_{\mathcal{D}_1 \cup \mathcal{D}_3} f^m_H(t, \delta_D(x))dx \le c(\varepsilon) t^{1/\alpha},$$

where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. From the (approximate) scaling property of the relativistic stable process, we have

$$t^{d/\alpha} \int_D f^m_H(t, \delta_D(x))dx = \int_D f^{mt}_H(1, \delta_D(x)/t^{1/\alpha})dx.$$

Now apply Lemmas 6.4.2 and 6.4.4 to the function $r \to f^{mt}_H(1,r)$ and we get for small enough t

$$\left| \int_D f^{mt}_H(1, \delta_D(x)/t^{1/\alpha})dx - C_2 \mathcal{H}^{d-1}(\partial D) t^{1/\alpha} \right| \le \varepsilon t^{1/\alpha}.$$

□

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