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Trace Estimates for Relativistic Stable Processes

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Abstract In this paper, we study the asymptotic behavior, as the time t goes to zero, of the trace of the semigroup of a killed relativistic α-stable process in bounded $C^{1,1}$ open sets and bounded Lipschitz open sets. More precisely, we establish the asymptotic expansion in terms of t of the trace with an error bound of order $t^{2/\alpha}t^{-d/\alpha}$ for $C^{1,1}$ open sets and of order $t^{1/\alpha}t^{-d/\alpha}$ for Lipschitz open sets. Compared with the corresponding expansions for stable processes, there are more terms between the orders $t^{-d/\alpha}$ and $t^{(2-d)/\alpha}$ for $C^{1,1}$ open sets, and, when $\alpha \in (0, 1]$, between the orders $t^{-d/\alpha}$ and $t^{(1-d)/\alpha}$ for Lipschitz open sets.

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1 Introduction and Statement of the Main Results

For any $m > 0$ and $\alpha \in (0, 2)$, a relativistic α-stable process X^m on $\mathbb{R}^d$ with mass m is a Lévy process with characteristic function given by

$$\mathbb{E}\left[\exp(i\xi \cdot (X_t^m - X_0^m))\right] = \exp(-t((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m)), \quad \xi \in \mathbb{R}^d. \tag{1.1}$$

The limiting case X^0, corresponding to $m = 0$, is a (rotationally) symmetric α-stable process on $\mathbb{R}^d$ which we will simply denote as X. The infinitesimal generator of X^m is $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$. Note that when $m = 1$, this infinitesimal generator reduces to $1 - (1 - \Delta)^{\alpha/2}$. Thus the 1-resolvent kernel of the relativistic α-stable process X^1 on $\mathbb{R}^d$ is

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just the Bessel potential kernel. When $\alpha = 1$, the infinitesimal generator reduces to the so-called free relativistic Hamiltonian $m - \sqrt{-\Delta + m^2}$. The operator $m - \sqrt{-\Delta + m^2}$ is very important in mathematical physics due to its application to relativistic quantum mechanics.

In this paper, we will be interested in the asymptotic behavior of the trace of the semigroup associated with killed relativistic α-stable processes in open sets of $\mathbb{R}^d$. The process X^m has a transition density $p^m(t,x,y) = p^m(t, y-x)$ given by the inverse Fourier transform

$$p^m(t,x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi x} e^{-t(|\xi|^2+m^{2/\alpha})^{\alpha/2}+mt} d\xi.$$

For any open set D in $\mathbb{R}^d$, the killed relativistic α-stable process $X_t^{m,D}$ is defined by

$$X_t^{m,D} = \begin{cases} X_t^m & \text{if } t < \tau_D^m, \\ \partial & \text{if } t \geq \tau_D^m, \end{cases}$$

where $\tau_D^m = \inf\{t > 0 : X_t^m \notin D\}$ is the first exit time of X^m from D. The process $X_t^{m,D}$ is a strong Markov process with a transition density $p_D^m(t,x,y)$ given by

$$p_D^m(t,x,y) = p^m(t,x,y) - r_D^m(t,x,y),$$

with

$$r_D^m(t,x,y) = \mathbb{E}_x \left[t > \tau_D^m;\, p^m(t - \tau_D^m, X^m_{\tau_D^m}, y) \right].$$

We denote by $(P_t^{m,D} : t \geq 0)$ the semigroup of X_t^m on $L^2(D)$: for any $f \in L^2(D)$,

$$P_t^{m,D} f(x) := \mathbb{E}_x \left[f(X_t^{m,D}) \right] = \int_D f(y) p_D^m(t,x,y) dy.$$

Whenever D is of finite volume, $P_t^{m,D}$ is a Hilbert-Schmidt operator mapping $L^2(D)$ into $L^\infty(D)$ for every $t > 0$. By general operator theory, there exist an orthonormal basis of eigenfunctions $\{\phi_n^{(m)}\}_{n=1}^\infty$ for $L^2(D)$ and corresponding eigenvalues $\{\lambda_n^{(m)}\}_{n=1}^\infty$ of the generator of the semigroup $P_D^{m,D}$ satisfying

$$0 < \lambda_1^{(m)} < \lambda_2^{(m)} \leq \lambda_3^{(m)} \leq \cdots$$

with $\lambda_n^{(m)} \to \infty$. By definition, we have

$$P_t^{m,D} \phi_n^{(m)}(x) = e^{-\lambda_n^{(m)} t} \phi_n^{(m)}(x), \quad x \in D, t > 0.$$

We also have

$$p_D^m(t,x,y) = \sum_{n=1}^\infty e^{-\lambda_n^{(m)} t} \phi_n^{(m)}(x) \phi_n^{(m)}(y).$$

$\lambda_n^{(0)}$ will be simply denoted by λ_n.

In the remainder of this paper, we assume $d \geq 2$. We are interested in finding the asymptotic behavior, as $t \to 0$, of the trace defined by

$$Z_D^m(t) = \int_D p_D^m(t,x,x) dx = \sum_{n=1}^\infty e^{-\lambda_n^{(m)} t} \int_D (\phi_n^{(m)})^2(x) dx = \sum_{n=1}^\infty e^{-\lambda_n^{(m)} t}.$$

It is shown in [2] that for any open set D of finite volume, it holds that

$$\lim_{t\to 0} t^{d/\alpha} Z_D^0 = C_1 |D|, \qquad C_1 = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}, \tag{1.2}$$

where $\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area of the unit sphere in $\mathbb{R}^d$. This is closely related to the growth of the eigenvalues of $P_t^{0,D}$: if $N^0(\lambda)$ is the number of eigenvalues λ_j such that $\lambda_j \le \lambda$, then it follows from the classical Karamata Tauberian theorem (see for example [10]) that

$$N^0(\lambda) \sim \frac{C_1|D|}{\Gamma(d/\alpha+1)}\lambda^{d/\alpha}, \quad \text{as } \lambda \to \infty. \tag{1.3}$$

This is the analogue for killed stable processes of the celebrated Weyl's asymptotic formula for the eigenvalues of the Dirichlet Laplacian. We will see later in this paper that exactly the same formula is true for relativistic stable processes. That is, the first term in the expansion of $Z_D^m(t)$ is the same as that of $Z_D^0(t)$ and Eq. 1.3 is also true for relativistic stable processes.

Our main goal in this paper is to get the asymptotic expansion of $Z_D^m(t)$ as $t \to 0$ under some additional assumptions on the smoothness of the boundary of D. Our work is inspired by the paper [7] for Brownian motion and the papers [2, 3] for stable processes. The first theorem is an asymptotic expansion of $Z_D^m(t)$ with error bound of order $t^{2/\alpha}t^{-d/\alpha}$ in $C^{1,1}$ open sets. To state the result precisely, we need some definitions. Recall that an open set D in $\mathbb{R}^d$ is said to be a (uniform) $C^{1,1}$ open set if there are (localization radius) $R > 0$ and Λ_0 such that for every $z \in \partial D$, there exist a $C^{1,1}$ function $\phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0, \cdots, 0) = 0$, $\nabla\phi(0) = (0, \ldots, 0)$, $|\nabla\phi(x) - \nabla\phi(y)| \le \Lambda_0|x - z|$ and an orthonormal coordinate system $CS_z : y = (y_1, \cdots, y_{d-1}, y_d) := (\tilde{y}, y_d)$ with origin at z such that $B(z, R) \cap D = \{y = (\tilde{y}, y_d) \in B(0, R) \text{ in } CS_z : y_d > \phi(\tilde{y})\}$. For $x \in \mathbb{R}^d$, let $\delta_D(x)$ denote the Euclidean distance between x and D^c and $\delta_{\partial D}(x)$ the Euclidean distance between x and ∂D. It is well known that a $C^{1,1}$ open set D satisfies both the *uniform interior ball condition* and the *uniform exterior ball condition*: there exists $r_0 < R$ such that for every $x \in D$ with $\delta_{\partial D}(x) \le r_0$ and $y \in \mathbb{R}^d \setminus \bar{D}$ with $\delta_{\partial D}(y) \le r_0$, there are $z_x, z_y \in \partial D$ so that $|x - z_x| = \delta_{\partial D}(x)$, $|y - z_y| = \delta_{\partial D}(y)$ and that $B(x_0, r_0) \subset D$ and $B(y_0, r_0) \subset \mathbb{R}^d \setminus \bar{D}$, where $x_0 = z_x + r_0(x - z_x)/|x - z_x|$ and $y_0 = z_y + r_0(y - z_y)/|y - z_y|$. In fact, D is a $C^{1,1}$ open set if and only if D satisfies the uniform interior ball condition and the uniform exterior ball condition (see [1, Lemma 2.2]). In this paper we call the pair (r_0, Λ_0) the characteristics of the $C^{1,1}$ open set D. For any open set D in $\mathbb{R}^d$, we use $|D|$ to denote the d-dimensional Lebesgue measure of D and $\mathcal{H}^{d-1}(\partial D)$ to denote the $(d-1)$-dimensional Hausdorff measure of ∂D. When D is a $C^{1,1}$ open set, $\mathcal{H}^{d-1}(\partial D)$ is equal to the surface measure $|\partial D|$ of ∂D. We will use H to denote the half space $\{x = (x_1, x_2, \cdots, x_d) : x_1 > 0\}$.

The following is the the first main result of this paper.

Theorem 1.1 *Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. Let k be the largest integer such that $k < \frac{2}{\alpha}$. Then the trace $Z_D^m(t)$ admits the following expansion*

$$Z_D^m(t) = C_1|D|t^{-\frac{d}{\alpha}} - C_2|\partial D|t^{\frac{1-d}{\alpha}} + \frac{\omega_d\Gamma(d/\alpha)|D|}{(2\pi)^d\alpha}t^{-\frac{d}{\alpha}}\sum_{n=1}^{k}\frac{m^n}{n!}t^n + O\left(\frac{t^{2/\alpha}}{t^{d/\alpha}}\right),$$

where C_1 is given in Eq. 1.2 and

$$C_2 = \int_0^\infty r_H^0(1, (r, \tilde{0}), (r, \tilde{0}))dr.$$

The second main result of the paper is an asymptotic expansion of $Z_D^m(t)$ with error bound of order $t^{1/\alpha}t^{-d/\alpha}$ in Lipschitz open sets. Before we state the second main result, we recall the definition of Lipschitz open sets. An open set D in $\mathbb{R}^d$ is called a Lipschitz open set if there exist constants R_0 (localization radius) and $\lambda > 0$ (Lipschitz constant) such that for every $z \in \partial D$ there exist a Lipschitz function $F : \mathbb{R}^{d-1} \to \mathbb{R}$ with Lipschitz constant λ and an orthornormal coordinate system $y = (y_1, \cdots, y_d)$ such that $D \cap B(z, R_0) = \{y : y_d > F(y_1, \cdots, y_{d-1})\} \cap B(z, R_0)$. Here is the second main result.

Theorem 1.2 *Suppose that D is a bounded Lipschitz open set in $\mathbb{R}^d$. Let j be the largest integer such that $j \le \frac{1}{\alpha}$. Then the trace $Z_D^m(t)$ admits the following expansion*

$$t^{d/\alpha} Z_D^m(t) = C_1|D| - C_2\mathcal{H}^{d-1}(\partial D)t^{1/\alpha} + \frac{\omega_d \Gamma(d/\alpha)|D|}{(2\pi)^d \alpha} \sum_{n=1}^{j} \frac{m^n}{n!} t^n + o(t^{1/\alpha}),$$

where C_1 and C_2 are the same as in Theorem 1.

The asymptotic behaviors of the trace $Z_D(t)$ of the killed Brownian motion (i.e., killed symmetric α-stable process with $\alpha = 2$) in bounded domains D of $\mathbb{R}^d$ have been extensively studied by many authors. It is shown in [5] that, when D is a bounded $C^{1,1}$ domain,

$$\left| Z_D(t) - (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| \right) \right| \le \frac{c|D|t^{1-d/2}}{R^2}, \quad t > 0.$$

The following asymptotic result

$$Z_D(t) = (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| + o(t^{1/2}) \right), \quad t \to 0, \tag{1.4}$$

was proved in [6] when D is a bounded C^1 domain. Equation 1.4 was subsequently extended to Lipschitz domains in [7].

The asymptotic behaviors of the trace $Z_D^0(t)$ of killed symmetric α-stable processes, $0 < \alpha < 2$, in open sets of $\mathbb{R}^d$ have been studied in [2, 3]. It was shown in [2] that, for any bounded $C^{1,1}$ open set D,

$$\left| Z_D^0(t) - \frac{C_1|D|}{t^{d/\alpha}} + \frac{C_2|\partial D|t^{1/\alpha}}{t^{d/\alpha}} \right| \le \frac{c|D|t^{2/\alpha}}{r_0^2 t^{d/\alpha}},$$

where C_1 and C_2 are the same as in Theorem 1 and c is a positive constant depending on d and α only. It was shown in [3] that, when D is a bounded Lipschitz domain, $Z_D^0(t)$ satisfies

$$t^{d/\alpha} Z_D^0(t) = C_1|D| - C_2\mathcal{H}^{d-1}(\partial D)t^{1/\alpha} + o(t^{1/\alpha}).$$

Remark 1.3 Note that the first term in the expansion of $Z_D^m(t)$ is exactly the same as in the case of $Z_D^0(t)$. However the rest of the terms are quite different. We note here that the coefficient of the term of order $t^{1/\alpha}t^{-d/\alpha}$ is the same in the stable process case, but in the case of relativistic stable processes for $C^{1,1}$ open sets, there are k intermediate terms of the form $t^k t^{-d/\alpha}$, where k is a positive integer less than $2/\alpha$. Since $0 < \alpha < 2$, there is at least one more term involved in the asymptotic expansion of $Z_D^m(t)$ than that of $Z_D^0(t)$ up to order of $t^{2/\alpha}t^{-d/\alpha}$. For Lipschitz open sets, when $\alpha \le 1$ there are j intermediate terms of the form $t^j t^{-d/\alpha}$, where j is an integer that is less than or equal to $1/\alpha$.

Remark 1.4 In [4], an asymptotic expansion for the trace of relativisitic α-stable processes in bounded $C^{1,1}$ open sets was established. Compared with Theorem 1, the expansion of [4] does not contain the intermediate terms.

The rest of the paper is organized as follows. In Section 2, we recall some basic facts about relativistic stable processes and present several preliminary results which will be used in Sections 3 and 4. Theorem 1 is proved in Section 3, while Theorem 2 is proved in Section 4.

Throughout this paper, we will use c to denote a positive constant depending (unless otherwise explicitly stated) only on d and α but whose value may change from line to line, even within a single line. In this paper, the big O notation $f(t) = O(g(t))$ always means that there exist constants C and $t_0 > 0$ such that $f(t) \le Cg(t)$ for all $0 < t < t_0$.

2 Preliminaries

In this section, we recall some basic facts about relativistic α-stable processes. From Eq 1.1, one can easily see that X^m has the following approximate scaling property:

$$\{m^{-1/\alpha}(X^1_{mt} - X^1_0), t \ge 0\} \text{ has the same law as } \{X^m_t - X^m_0, t \ge 0\}.$$

In terms of transition densities, this approximate scaling property can be written as

$$p^m(t, x, y) = m^{d/\alpha} p^1(mt, m^{1/\alpha}x, m^{1/\alpha}y). \tag{2.1}$$

It is well known that the transition density $p^m_D(t, x, y)$ of $X^{m,D}$ is continuous on $(0, \infty) \times D \times D$. Since both $p^m(t, x, y)$ and $p^m_D(t, x, y)$ are continuous on $(0, \infty) \times D \times D$, $r^m_D(t, x, y) = p^m(t, x, y) - p^m_D(t, x, y)$ is also continuous there. $p^m_D(t, x, y)$ and $r^m_D(t, x, y)$ also enjoy the following approximate scaling property:

$$p^1_{m^{1/\alpha}D}(t, x, y) = m^{-d/\alpha} p^m_D(t/m, x/m^{1/\alpha}, y/m^{1/\alpha}), \tag{2.2}$$

$$r^1_{m^{1/\alpha}D}(t, x, y) = m^{-d/\alpha} r^m_D(t/m, x/m^{1/\alpha}, y/m^{1/\alpha}). \tag{2.3}$$

The Lévy measure of the relativistic α-stable process X^m has a density

$$J^m(x) = j^m(|x|) := \frac{\alpha}{2\Gamma(1-\alpha/2)} \int_0^\infty (4\pi u)^{-d/2} e^{-|x|^2/4u} e^{-m^{2/\alpha}u} u^{-(1+\alpha/2)} du,$$

which is continuous and radially decreasing on $\mathbb{R}^d \setminus \{0\}$ (see [13, Lemma 2]). Put $J^m(x, y) := j^m(|x-y|)$. Let $\mathcal{A}(d, -\alpha) := \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma(\frac{d+\alpha}{2})\Gamma(1-\frac{\alpha}{2})^{-1}$. Using change of variables twice, first with $u = |x|^2 v$ then with $v = 1/s$, we get

$$J^m(x, y) = \mathcal{A}(d, -\alpha)|x-y|^{-d-\alpha}\psi(m^{1/\alpha}|x-y|), \tag{2.4}$$

where

$$\psi(r) := 2^{-(d+\alpha)}\Gamma\left(\frac{d+\alpha}{2}\right)^{-1} \int_0^\infty s^{(d+\alpha)/2-1} e^{-s/4-r^2/s} ds, \tag{2.5}$$

which satisfies $\psi(0) = 1$ and

$$c_1^{-1} e^{-r} r^{(d+\alpha-1)/2} \le \psi(r) \le c_1 e^{-r} r^{(d+\alpha-1)/2} \quad \text{on } [1, \infty)$$

for some $c_1 > 1$ (see [9, pp. 276-277] for details). We denote the Lévy density of X by

$$J(x, y) := J^0(x, y) = \mathcal{A}(d, -\alpha)|x-y|^{-d-\alpha}.$$

Note that from Eqs. 2.4 and 2.5 we see that for any $x \in \mathbb{R}^d \setminus \{0\}$

$$j^m(|x|) \le j^0(|x|).$$

It follows from [8, Theorem 4.1.] that, for any positive constants M and T there exists a constant $c > 1$ such that for all $m \in (0, M]$, $t \in (0, T]$, and $x, y \in \mathbb{R}^d$ we have

$$c^{-1}\left(t^{-d/\alpha} \wedge tJ^m(x, y)\right) \le p^m(t, x, y) \le c\left(t^{-d/\alpha} \wedge tJ^m(x, y)\right). \tag{2.6}$$

We will need a simple lemma from [11] about the relationship between $r_D^m(t, x, y)$ and $r_D^0(t, x, y)$. The lemma is true in much more general situations but we just need it when one of the processes is a symmetric α-stable process and the other is a relativistic α-stable process.

Lemma 2.1 *Suppose that X and Y are two Lévy processes with Lévy densities J^X and J^Y, respectively. Suppose that $\sigma = J^X - J^Y$ is nonnegative on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} \sigma(x)dx = \ell < \infty$ and D is an open set. Then for any $x \in D$ and $t > 0$,*

$$p_D^Y(t, x, \cdot) \le e^{\ell t} p_D^X(t, x, \cdot) \quad a.s.$$

If, in addition, $p^X(t, \cdot)$ and $p^Y(t, \cdot)$ are continuous, then we have for $x, y \in D$,

$$r_D^Y(t, x, y) \le e^{2\ell t} r_D^X(t, x, y).$$

The next proposition is the (generalized) Ikeda-Watanabe formula for the relativistic stable process, which describes the joint distribution of τ_D^m and $X^m_{\tau_D^m}$.

Proposition 2.2 (Proposition 2.7 [12]) *Assume that D is an open subset of $\mathbb{R}^d$ and A is a Borel set such that $A \subset D^c \setminus \partial D$. If $0 \le t_1 < t_2 < \infty$, then*

$$\mathbb{P}_x\left(X^m_{\tau_D^m} \in A,\ t_1 < \tau_D^m < t_2\right) = \int_D \int_{t_1}^{t_2} p_D^m(s, x, y)ds \int_A J^m(y, z)dzdy, \quad x \in D.$$

Now we state a simple lemma about the upper bound of $r_D^m(t, x, y)$, which is an analogue of [2, Lemma 2.1] for stable processes.

Lemma 2.3 *Let M, T be positive constants. Then there exists a constant $c = c(d, \alpha, M, T)$ such that for all $m \in (0, M]$ and $t \in (0, T]$ we have*

$$r_D^m(t, x, y) \le c\left(t^{-d/\alpha} \wedge \frac{t\psi(m^{1/\alpha}\delta_D(x))}{\delta_D(x)^{d+\alpha}}\right).$$

Proof Since ψ is eventually decreasing and $\psi(0) = 1 > 0$, there exists a constant $c_1 > 0$ such that $\psi(x) \le c_1\psi(y)$ for all $0 \le y \le x$. Now from the definition of $r_D^m(t, x, y)$ and Eq. 2.6 we have

$$\begin{aligned} r_D^m(t, x, y) &= r_D^m(t, y, x) \\ &\le \mathbb{E}_y\left[t > \tau_D^m;\ p^m(t - \tau_D^m, X^m_{\tau_D^m}, x)\right] \\ &\le \mathbb{E}_y\left[c\left(t^{-d/\alpha} \wedge \frac{t\psi(m^{1/\alpha}|x - X^m_{\tau_D^m}|)}{|x - X^m_{\tau_D^m}|^{d+\alpha}}\right)\right] \\ &\le cc_1\left(t^{-d/\alpha} \wedge \frac{t\psi(m^{1/\alpha}\delta_D(x))}{\delta_D(x)^{d+\alpha}}\right). \end{aligned}$$

□

We will need two results from [2]. The first result is about the difference $p^m_F(t,x,y) - p^m_D(t,x,y)$ when $D \subset F$. The proof in [2], given for stable processes, mainly uses the strong Markov property and it works for all strong Markov processes with transition densities.

Proposition 2.4 (Proposition 2.3 [2]) *Let D and F be open sets in $\mathbb{R}^d$ such that $D \subset F$. Then for any $x, y \in \mathbb{R}^d$ we have*

$$p^m_F(t,x,y) - p^m_D(t,x,y) = \mathbb{E}_x\left[\tau^m_D < t, X^m_{\tau^m_D} \in F \setminus D : p^m_F(t-\tau^m_D, X^m_{\tau^m_D}, y)\right].$$

Now we introduce some notation. Recall that if D is a $C^{1,1}$ open set with characteristics (r_0, Λ_0), then for any point $y \in \partial D$ there are two open balls B_1 and B_2 with radii r_0 such that $B_1 \subset D$, $B_2 \subset \mathbb{R}^d \setminus (D \cup \partial D)$, $\partial B_1 \cap \partial B_2 = \{y\}$. For any $x \in D$ with $\delta_D(x) < r_0/2$ there exists a unique point $z_x \in \partial D$ such that $\delta_D(x) = |x - z_x|$. Let $B_1 = B(z_1, r_0)$, $B_2 = B(z_2, r_0)$ be the balls for the point z_x. Let $H(x)$ be the half-space containing B_1 such that $\partial H(x)$ contains z_x and is perpendicular to the segment $\overline{z_1 z_2}$. The next proposition says that, in case of the symmetric α-stable process, for small t, the quantity $r^0_D(t,x,x)$ can be replaced by $r^0_{H(x)}(t,x,x)$, which was a very crucial step in proving the main result in [2].

Proposition 2.5 (Proposition 3.1 of [2]) *Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ open set with characteristics (r_0, Λ_0). Then, for any x with $\delta_{\partial D}(x) < r_0/2$ and $t > 0$ with $t^{1/\alpha} \le r_0/2$, we have*

$$\left| r^0_D(t,x,x) - r^0_{H(x)}(t,x,x) \right| \le \frac{c t^{1/\alpha}}{r_0 t^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_{\partial D}(x)} \right)^{d+\frac{\alpha}{2}-1} \wedge 1 \right).$$

We will need some facts about the "stability" of the surface area of the boundary of $C^{1,1}$ open sets. The following lemma is [5, Lemma 5].

Lemma 2.6 *Let D be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristic (r_0, Λ_0) and define for $0 \le q < r_0$,*

$$D_q = \{x \in D : \delta_D(x) > q\}.$$

Then

$$\left(\frac{r_0 - q}{r_0}\right)^{d-1} |\partial D| \le |\partial D_q| \le \left(\frac{r_0}{r_0 - q}\right)^{d-1} |\partial D|, \quad 0 \le q < r_0.$$

The following result is [2, Corollary 2.14].

Lemma 2.7 *Let D be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristic (r_0, Λ_0). For any $0 < q \le r_0/2$, we have*

1. $2^{-d+1}|\partial D| \le |\partial D_q| \le 2^{d-1}|\partial D|$,
2. $|\partial D| \le \frac{2^d |D|}{r_0}$,
3. $\big||\partial D_q| - |\partial D|\big| \le \frac{2^d dq |\partial D|}{r_0} \le \frac{2^{2d} dq |D|}{r_0^2}$.

3 Proof of Theorem 1.1

We first prove that $\lim_{t \to 0} t^{\frac{d}{\alpha}} Z^m_D(t)$ exists and identify the limit.

Lemma 3.1 *The limit* $\lim_{t\to 0} t^{\frac{d}{\alpha}} Z_D^m(t)$ *exists and is equal to* $C_1|D|$, *where* C_1 *is the constant in Theorem 1.1.*

Proof By definition,

$$t^{d/\alpha} Z_D^m(t) = t^{d/\alpha} \int_D p_D^m(t,x,x)dx$$
$$= t^{d/\alpha} \left(\int_D p^m(t,x,x)dx - \int_D r_D^m(t,x,x)dx \right). \tag{3.1}$$

For the first integral on the right hand side of Eq. 3.1, note that, by the approximate scaling property (2.1) and the dominated convergence theorem, we have, as $t \to 0$,

$$t^{d/\alpha} \left(\int_D p^m(t,x,x)dx \right) = \int_D p^{tm}(1, x/t^{1/\alpha}, x/t^{1/\alpha})dx = |D| p^{tm}(1,0)$$
$$\to |D| \cdot p^0(1,0) = |D| \cdot \frac{\Gamma(d/\alpha)\omega_d}{(2\pi)^d \alpha}.$$

It remains to show that $\lim_{t\to 0} t^{d/\alpha} \int_D r_D^m(t,x,x)dx = 0$. By Lemma 2.3 we have that

$$t^{d/\alpha} r_D^m(t,x,y) \le c, \qquad (t,x,y) \in (0,1] \times D \times D,$$

for some $c > 0$. Hence we have by the dominated convergence theorem,

$$t^{d/\alpha} \int_{D \setminus D_{t^{1/2\alpha}}} r_D^m(t,x,x) \to 0 \quad \text{as } t \to 0.$$

For $x \in D_{t^{1/2\alpha}}$ we have by Lemma 2.3 again for $t \in (0,1]$,

$$r_D^m(t,x,x) \le c\, t^{\frac{1}{2} - \frac{d}{2\alpha}}, \qquad x \in D_{t^{1/2\alpha}}.$$

Hence $\lim_{t\to 0} t^{d/\alpha} \int_{D_{t^{1/2\alpha}}} r_D^m(t,x,x)dx = 0$. □

It follows from Lemma 3.1 that if $N^m(\lambda)$ denotes the number of eigenvalues $\lambda_j^{(m)}$ such that $\lambda_j^{(m)} \le \lambda$, then it follows from the classical Karamata Tauberian theorem (see for example [10]) that

$$N^m(\lambda) \sim \frac{C_1|D|}{\Gamma(d/\alpha + 1)} \lambda^{d/\alpha}, \quad \text{as } \lambda \to \infty.$$

This is the analogue for killed relativistic stable processes of the celebrated Weyl's asymptotic formula for the eigenvalues of the Dirichlet Laplacian and it is already proved in [4] (see [4, (1.10)]). This result has been known at least since 2009, see [4, Remark 1.2].

Now we focus on identifying the next terms in $Z_D^m(t)$. For this, we need to find the order of t in $Z_D^m(t) - C_1 t^{-\frac{d}{\alpha}}$. Note that by Lemma 3.1,

$$Z_D^m(t) - C_1 t^{-d/\alpha} = \int_D p_D^m(t,x,x) - p^0(t,x,x)dx$$
$$= \int_D \left(p^m(t,x,x) - p^0(t,x,x) \right) dx - \int_D r_D^m(t,x,x)dx.$$

The next lemma gives the orders of t in $p^m(t,x,x) - p^0(t,x,x)$ up to $t^{\frac{2}{\alpha}} t^{-\frac{d}{\alpha}}$.

Lemma 3.2 *Let k be the largest integer such that $k < \frac{2}{\alpha}$. Then we have*

$$p^m(t,x,x) - p^0(t,x,x) = t^{-d/\alpha} \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha} \sum_{n=1}^{k} \frac{m^n}{n!} t^n + O(t^{2/\alpha} t^{-d/\alpha}).$$

Proof By the scaling property (2.1) we have

$$\begin{aligned} &p^m(t,x,x) - p^0(t,x,x) = p^m(t,0) - p^0(t,0) \\ &= t^{-d/\alpha} \left(p^{tm}(1,0) - p^0(1,0) \right) \\ &= t^{-d/\alpha} (2\pi)^{-d} \int_{\mathbb{R}^d} \left(e^{-(|\xi|^2 + (mt)^{2/\alpha})^{\alpha/2} + mt} - e^{-|\xi|^\alpha} \right) d\xi. \end{aligned}$$

Note that for any $x \geq 0$ we have $(1+x)^{\alpha/2} \leq 1 + \frac{\alpha}{2} x$. Thus

$$\left(|\xi|^2 + (mt)^{2/\alpha} \right)^{\alpha/2} = |\xi|^\alpha \left(1 + \frac{(mt)^{2/\alpha}}{|\xi|^2} \right)^{\alpha/2} \leq |\xi|^\alpha \left(1 + \frac{\alpha}{2} \frac{(mt)^{2/\alpha}}{|\xi|^2} \right).$$

Consequently

$$\begin{aligned} &0 \leq e^{-|\xi|^\alpha} - e^{-\left(|\xi|^2 + (mt)^{2/\alpha}\right)^{\alpha/2}} \\ &\leq e^{-|\xi|^\alpha} - e^{-|\xi|^\alpha \left(1 + \frac{\alpha}{2} \frac{(mt)^{2/\alpha}}{|\xi|^2}\right)} = e^{-|\xi|^\alpha} \left(1 - e^{-\frac{\alpha}{2} \frac{(mt)^{2/\alpha}}{|\xi|^{2-\alpha}}} \right) \\ &\leq e^{-|\xi|^\alpha} \left(\frac{\alpha}{2} \frac{(mt)^{2/\alpha}}{|\xi|^{2-\alpha}} \right), \end{aligned}$$

where we used $1 - e^{-x} \leq x$ for all $x \geq 0$ in the last inequality above. Therefore

$$\begin{aligned} &0 \leq \int_{\mathbb{R}^d} e^{-\left(|\xi|^2 + (mt)^{2/\alpha}\right)^{\alpha/2} + mt} - e^{-|\xi|^\alpha} d\xi \\ &\leq \int_{\mathbb{R}^d} \left| e^{-\left(|\xi|^2 + (mt)^{2/\alpha}\right)^{\alpha/2} + mt} - e^{-|\xi|^\alpha} e^{mt} + e^{-|\xi|^\alpha} e^{mt} - e^{-|\xi|^\alpha} \right| d\xi \\ &\leq \int_{\mathbb{R}^d} \left| e^{-\left(|\xi|^2 + (mt)^{2/\alpha}\right)^{\alpha/2} + mt} - e^{-|\xi|^\alpha} e^{mt} \right| d\xi + \int_{\mathbb{R}^d} \left| e^{-|\xi|^\alpha} e^{mt} - e^{-|\xi|^\alpha} \right| d\xi \\ &\leq \int_{\mathbb{R}^d} e^{mt} e^{-|\xi|^\alpha} \left(\frac{\alpha}{2} \frac{(mt)^{2/\alpha}}{|\xi|^{2-\alpha}} \right) d\xi + \int_{\mathbb{R}^d} e^{-|\xi|^\alpha} \left(e^{mt} - 1 \right) d\xi \\ &= e^{mt} \frac{\alpha}{2} (mt)^{2/\alpha} \int_{\mathbb{R}^d} \frac{e^{-|\xi|^\alpha}}{|\xi|^{2-\alpha}} d\xi + \sum_{n=1}^{\infty} \frac{(mt)^n}{n!} \int_{\mathbb{R}^d} e^{-|\xi|^\alpha} d\xi. \end{aligned}$$

Since $k + j \geq 2/\alpha$ for any $j \geq 1$, we have $\displaystyle\sum_{n=k+1}^{\infty} \frac{(mt)^n}{n!} = O(t^{2/\alpha})$. Therefore

$$\int_{\mathbb{R}^d} \left(e^{-\left(|\xi|^2 + (mt)^{2/\alpha}\right)^{\alpha/2} + mt} - e^{-|\xi|^\alpha} \right) d\xi = O(t^{2/\alpha}) + \frac{\omega_d \Gamma(d/\alpha)}{\alpha} \sum_{n=1}^{k} \frac{(mt)^n}{n!} \qquad (3.2)$$

and

$$p^m(t,x,x) - p^0(t,x,x) = t^{-d/\alpha} \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha} \sum_{n=1}^{k} \frac{m^n}{n!} t^n + O(t^{2/\alpha} t^{-d/\alpha}). \tag{3.3}$$

□

Now we try to find the orders of t in the expansion of $\int_D r_D^m(t,x,x)dx$ up to the order of $t^{\frac{2}{\alpha}} t^{-\frac{d}{\alpha}}$. For this, we need to assume some regularity condition on the boundary of D. Hence in the remainder of this section we assume that D is a bounded $C^{1,1}$ open set with characteristic (r_0, Λ_0). We also assume that $t^{1/\alpha} \le \frac{r_0}{2}$.

We first deal with the contribution in $D_{r_0/2}$.

Lemma 3.3 *There exists* $c = c(d,\alpha) > 0$ *such that*

$$\int_{D_{r_0/2}} r_D^m(t,x,x)dx \le ce^{2mt} \frac{|D| t^{2/\alpha}}{r_0^2 t^{d/\alpha}}.$$

Proof It follows from [13, Lemma 2] and Lemma 2.1 that $r_D^m(t,x,y) \le e^{2mt} r_D^0(t,x,y)$. By [2, (3.2)] we know that

$$\int_{D_{r_0/2}} r_D^0(t,x,y)dx \le \frac{c|D|t^{2/\alpha}}{r_0^2 t^{d/\alpha}}. \tag{3.4}$$

The desired assertion follows immediately. □

Lemma 3.4 *There exists* $c = c(d,\alpha) > 0$ *such that*

$$r_D^m(t,x,x) - r_{H(x)}^m(t,x,x) \le ce^{2mt} \frac{t^{1/\alpha}}{t^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\frac{\alpha}{2}-1} \wedge 1 \right)$$

and

$$\int_{D \setminus D_{r_0/2}} \left(r_D^m(t,x,x) - r_{H(x)}^m(t,x,x) \right) dx \le ce^{2mt} \frac{t^{2/\alpha}}{t^{d/\alpha}}.$$

Proof If the first assertion of the lemma is right, then it is easy to see that

$$\int_{D \setminus D_{r_0/2}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\frac{\alpha}{2}-1} \wedge 1 \right) dx \le ct^{1/\alpha}.$$

Hence we focus on proving the first assertion. By [2, (3.4)], we know that

$$r_D^0(t,x,x) - r_{H(x)}^0(t,x,x) \le c \frac{t^{1/\alpha}}{t^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\frac{\alpha}{2}-1} \wedge 1 \right).$$

Recall that $J^m(x) \le J^0(x)$ for any $x \in \mathbb{R}^d \setminus \{0\}$. Now it follows from the generalized Ikeda-Watanabe formula (Proposition 2.2), Proposition 2.4, and Lemma 2.1 that

$$\begin{aligned}
& r^m_D(t,x,x) - r^m_{H(x)}(t,x,x) \\
&= \mathbb{E}_x \left[t > \tau^m_D, X^m_{\tau^m_D} \in H(x)\setminus D;\, p^m_{H(x)}(t-\tau^m_D, X^m_{\tau^m_D}, x) \right] \\
&= \int_D \int_0^t p^m_D(s,x,y)ds \int_{H(x)\setminus D} J^m(y,z) p^m_{H(x)}(t-s,z,x)dzdy \\
&\le e^{2mt} \int_D \int_0^t p^0_D(s,x,y)ds \int_{H(x)\setminus D} J^0(y,z) p^0_{H(x)}(t-s,z,x)dzdy \\
&= e^{2mt}\mathbb{E}_x \left[t > \tau^0_D, X_{\tau^0_D} \in H(x)\setminus D;\, p^0_{H(x)}(t-\tau^0_D, X_{\tau^0_D}, x) \right] \\
&= e^{2mt} \left(r^0_D(t,x,x) - r^0_{H(x)}(t,x,x) \right) \\
&\le ce^{2mt} \frac{t^{1/\alpha}}{t^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)}\right)^{d+\frac{\alpha}{2}-1} \wedge 1 \right).
\end{aligned}$$

For convenience, we define the following notation.

$$f^m_H(t,r) := r^m_H(t,(r,\tilde{0}),(r,\tilde{0})), \qquad r > 0.$$

Lemma 3.5 *There exists $c = c(d,\alpha) > 0$ such that*

$$\int_{D\setminus D_{r_0/2}} r^m_{H(x)}(t,x,x)dx - t^{1/\alpha}t^{-d/\alpha} \int_0^{\frac{r_0}{2t^{1/\alpha}}} |\partial D| f^{tm}_H(1,u)du \le ct^{2/\alpha}t^{-d/\alpha}.$$

□

Proof Using the scaling relation (2.3) we get

$$\begin{aligned}
& \int_{D\setminus D_{r_0/2}} r^m_{H(x)}(t,x,x)dx \\
&= \int_0^{r_0/2} |\partial D_u| f^m_H(t,u)du \\
&= \int_0^{r_0/2} |\partial D_u| t^{-d/\alpha} f^{tm}_H(1,u/t^{1/\alpha})du \\
&= t^{1/\alpha}t^{-d/\alpha} \int_0^{r_0/2t^{1/\alpha}} |\partial D_{ut^{1/\alpha}}| f^{tm}_H(1,u)du.
\end{aligned}$$

It follows from Lemma 2.7 that $\left| |\partial D_q| - |\partial D| \right| \le \frac{2^d dq|\partial D|}{r_0} \le \frac{2^{2d} dq|D|}{r_0^2}$ for any $q \le r_0/2$. Hence

$$\begin{aligned}
& \left| \int_{D\setminus D_{r_0/2}} r^m_{H(x)}(t,x,x) - t^{1/\alpha}t^{-d/\alpha} \int_0^{\frac{r_0}{2t^{1/\alpha}}} |\partial D| f^{tm}_H(1,u)du \right| \\
&\le t^{1/\alpha}t^{-d/\alpha} \int_0^{\frac{r_0}{2t^{1/\alpha}}} \left| |\partial D_{ut^{1/\alpha}}| - |\partial D| \right| f^{tm}_H(1,u)du \\
&\le c_1 t^{2/\alpha}t^{-d/\alpha} \int_0^\infty u f^{tm}_H(1,u)du \\
&\le c_2 t^{2/\alpha}t^{-d/\alpha}.
\end{aligned}$$

Lemma 3.6 *There exists $c = c(d, \alpha) > 0$ such that* □

$$t^{1/\alpha} t^{-d/\alpha} \int_0^\infty |\partial D| f_H^{tm}(1, u) du - t^{1/\alpha} t^{-d/\alpha} \int_0^{\frac{r_0}{2t^{1/\alpha}}} |\partial D| f_H^{tm}(1, u) du \le c t^{2/\alpha} t^{-d/\alpha}.$$

Proof It follows from Lemma 2.1 that

$$\begin{aligned}
&t^{1/\alpha} t^{-d/\alpha} \int_0^\infty |\partial D| f_H^{tm}(1, u) du - t^{1/\alpha} t^{-d/\alpha} \int_0^{\frac{r_0}{2t^{1/\alpha}}} |\partial D| f_H^{tm}(1, u) du \\
&= t^{1/\alpha} t^{-d/\alpha} \int_{\frac{r_0}{2t^{1/\alpha}}}^\infty |\partial D| f_H^{tm}(1, u) du \\
&= t^{1/\alpha} t^{-d/\alpha} |\partial D| \int_{\frac{r_0}{2t^{1/\alpha}}}^\infty f_H^{tm}(1, u) du \\
&\le e^{2mt} t^{1/\alpha} t^{-d/\alpha} |\partial D| \int_{\frac{r_0}{2t^{1/\alpha}}}^\infty f_H^0(1, u) du.
\end{aligned}$$

For $q \ge r_0/(2t^{1/\alpha})$ we have $f_H^0(1, q) \le c q^{-d-\alpha} \le c q^{-2}$. Hence

$$\int_{\frac{r_0}{2t^{1/\alpha}}}^\infty f_H^0(1, u) du \le c \int_{\frac{r_0}{2t^{1/\alpha}}}^\infty \frac{dq}{q^2} \le c \frac{t^{1/\alpha}}{r_0}$$

and the result now follows. □

Lemma 3.7 $\lim_{t \downarrow 0} \int_0^\infty f_H^{tm}(1, u) du = \int_0^\infty f_H^0(1, u) du.$

Proof This follows immediately from the continuity of $m \mapsto r_D^m(t, x, y)$ and the dominated convergence theorem. □

Proof of Theorem 1.1 Combining Lemmas 3.1-3.7, we immediately arrive at Theorem 1.1.

4 Proof of Theorem 1.2

In this section we always assume that D is a bounded Lipschitz open set in $\mathbb{R}^d$. The argument of this section is similar to previous section and [3]. We will follow the argument in [3] closely, making necessary modifications for relativistic stable processes. Note that even though the main theorem in [3] is stated for a Lipschitz domain, it remains true for a bounded Lipschitz open set.

First we need two technical facts which play crucial roles later. The first proposition is [3, Proposition 2.9] and we will state it here for reader's convenience.

Proposition 4.1 (Proposition 2.9. [3]) *Suppose that $f : (0, \infty) \to [0, \infty)$ is continuous and satisfies $f(r) \le c(1 \wedge r^{-\beta})$ for some $\beta > 1$. Furthermore, suppose that for any*

$0 < R_1 < R_2 < \infty$, *f is Lipschitz on* $[R_1, R_2]$. *Then we have*

$$\lim_{\eta\to 0^+} \frac{1}{\eta}\int_D f\left(\frac{\delta_D(x)}{\eta}\right)dx = \mathcal{H}^{d-1}(\partial D)\int_0^\infty f(r)dr.$$

Lemma 4.2 *Suppose that* $f:(0,\infty)\to[0,\infty)$ *is continuous and satisfies* $f(r)\le c_1(1\wedge r^{-\beta})$ *for some* $\beta > 1$. *Furthermore, suppose that for any* $0 < R_1 < R_2 < \infty$, *f is Lipschitz on* $[R_1, R_2]$. *Let* $\{f^\eta : \eta > 0\}$ *be continuous functions from* $(0,\infty)$ *to* $[0,\infty)$ *such that, for any* $0 < L < M < \infty$, $\lim_{\eta\to 0} f^\eta(r) = f(r)$ *uniformly for* $r \in [L, M]$. *Suppose that there exists* $c_2 > 0$ *such that* $f^\eta(r)\le c_2 f(r)$ *for all* $\eta\le 1$. *Then we have*

$$\lim_{\eta\to 0^+} \frac{1}{\eta}\int_D f^\eta\left(\frac{\delta_D(x)}{\eta}\right)dx = \mathcal{H}^{d-1}(\partial D)\int_0^\infty f(r)dr.$$

Proof Let $\psi_\eta(r) = \eta^{-1}|\{x\in D : \delta_D(x) < \eta r\}|$. Note (cf. proof of [7, Proposition 1.1]) that $\psi_\eta(r)\le c$ for all $\eta, r > 0$ and that

$$\eta^{-1}\int_D f\left(\frac{\delta_D(x)}{\eta}\right)dx = \int_0^\infty f(r)d\psi_\eta(r),$$

and

$$\eta^{-1}\int_D f^\eta\left(\frac{\delta_D(x)}{\eta}\right)dx = \int_0^\infty f^\eta(r)d\psi_\eta(r).$$

It was shown in [3, Proposition 2.9.] that, for any $0 < R_1 < R_2 < \infty$ and $\eta > 0$, f satisfies

$$\int_0^{R_1} f(r)d\psi_\eta(r)\le cR_1, \tag{4.1}$$

$$\int_{R_2}^\infty f(r)d\psi_\eta(r)\le c\eta^{\beta-1} + cR_2^{1-\beta}, \tag{4.2}$$

$$\lim_{\eta\to 0^+}\int_{R_1}^{R_2} f(r)d\psi_\eta(r) = \mathcal{H}^{d-1}(\partial D)\int_{R_1}^{R_2} f(r)dr.$$

Since $f^\eta\le c_2 f$ for $\eta\le 1$ we have the same inequalities as Eqs. 4.1 and 4.2 for f^η, $\eta\le 1$. Hence it is enough to show that

$$\lim_{\eta\to 0^+}\int_{R_1}^{R_2} f^\eta(r)d\psi_\eta(r) = \mathcal{H}^{d-1}(\partial D)\int_{R_1}^{R_2} f(r)dr.$$

For any partition $R_1 = x_0 < x_1 < \cdots < x_n = R_2$ of $[R_1, R_2]$, we have

$$\begin{aligned}
&\left|\sum_{i=1}^n f^\eta(x_i)\left(\psi_n(x_i)-\psi_n(x_{i-1})\right) - \sum_{i=1}^n f(x_i)\left(\psi_n(x_i)-\psi_n(x_{i-1})\right)\right| \\
&= \sum_{i=1}^n |f^\eta(x_i) - f(x_i)|\left(\psi_n(x_i)-\psi_n(x_{i-1})\right) \\
&\le \| f^\eta - f \|_{L^\infty[R_1,R_2]}\,\psi_\eta(R_2).
\end{aligned}$$

Note that for any $\eta > 0$ the function $r\to\psi_\eta(r)$ is nondecreasing and for any $\eta > 0$, $r > 0$ we have $\psi_\eta(r)\le cr$ for some constant c. Since $f^\eta\to f$ uniformly on $r\in[R_1, R_2]$, taking supremum for all possible partitions gives

$$\lim_{\eta\to 0^+}\int_{R_1}^{R_2} f^\eta(r)d\psi_\eta(r) = \lim_{\eta\to 0^+}\int_{R_1}^{R_2} f(r)d\psi_\eta(r) = \mathcal{H}^{d-1}(\partial D)\int_{R_1}^{R_2} f(r)dr.$$

Lemma 4.3 *For any $0 < L < M < \infty$, $p^m(t, x, y)$ converges uniformly to $p^0(t, x, y)$ as $m \to 0$ for $(t, x, y) \in [L, M] \times \mathbb{R}^d \times \mathbb{R}^d$.*

Proof Note that

$$\left|p^m(t, x, y)-p^0(t, x, y)\right| = \left|(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi(y-x)} \left(e^{-t((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m)} - e^{-t|\xi|^\alpha}\right) d\xi\right|$$
$$\le (2\pi)^{-d} \int_{\mathbb{R}^d} \left|e^{-i\xi(y-x)} \left(e^{-t((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m)} - e^{-t|\xi|^\alpha}\right)\right| d\xi$$
$$\le (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m)} - e^{-t|\xi|^\alpha} d\xi$$
$$= (2\pi)^{-d}(p^m(t, 0) - p^0(t, 0)).$$

Now it follows from the proof of Lemma 3.2 that for $t \in [L, M]$ and $x, y \in \mathbb{R}^d$,

$$\left|p^0(t, x, y) - p^m(t, x, y)\right|$$
$$\le t^{-d/\alpha}(2\pi)^{-d} \left(e^{mt}\frac{\alpha}{2}(mt)^{2/\alpha} \int_{\mathbb{R}^d} \frac{e^{-|\xi|^\alpha}}{|\xi|^{2-\alpha}} d\xi + \sum_{n=1}^{\infty} \frac{(mt)^n}{n!} \int_{\mathbb{R}^d} e^{-|\xi|^\alpha} d\xi\right)$$
$$\le L^{-d/\alpha}(2\pi)^{-d} \left(e^{mM}\frac{\alpha}{2}(mM)^{2/\alpha} \int_{\mathbb{R}^d} \frac{e^{-|\xi|^\alpha}}{|\xi|^{2-\alpha}} d\xi + \sum_{n=1}^{\infty} \frac{(mM)^n}{n!} \int_{\mathbb{R}^d} e^{-|\xi|^\alpha} d\xi\right).$$

The last quantity above converges to 0 as $m \to 0$. □

Lemma 4.4 *For any $0 < L < M < \infty$ and $m > 0$,*

$$\lim_{t\to 0} f_H^{tm}(1, r) = f_H^0(1, r), \quad \text{uniformly in } r \in [L, M],$$

that is, given $\varepsilon > 0$ there exists $t_0 > 0$ such that for $0 \le t \le t_0$ we have

$$\sup_{r\in[L,M]} \left|r_H^{tm}(1, (r, \tilde{0}), (r, \tilde{0})) - r_H^0(1, (r, \tilde{0}), (r, \tilde{0}))\right| < \varepsilon.$$

Proof Recall that $r_H^0(t, x, y) = \mathbb{E}_x[\tau_H^0 < t, p^0(t - \tau_H^0, X_{\tau_H^0}, y)]$ and $r_H^m(t, x, y) = \mathbb{E}_x[\tau_H^m < t, p^m(t - \tau_H^m, X^m_{\tau_H^m}, y)]$. It is well known that

$$p^0(t, x, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}.$$

Since $|X^0_{\tau_H^0} - (r, \tilde{0})| > L$, we have, together with Lemma 2.1,

$$p^0(1 - \tau_H^0, X^0_{\tau_H^0}, (r, \tilde{0})) \le c\frac{1 - \tau_H^0}{L^{d+\alpha}},$$

$$p^{tm}(1 - \tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r, \tilde{0})) \le ce^{tm}\frac{1 - \tau_H^{tm}}{L^{d+\alpha}}.$$

Now take δ_1 small so that

$$\mathbb{E}_{(r,\tilde{0})}\left[1 - \delta_1 \le \tau_H^0 < 1, p^0(1 - \tau_H^0, X^0_{\tau_H^0}, (r, \tilde{0}))\right] < \varepsilon, \tag{4.3}$$

$$\mathbb{E}_{(r,\tilde{0})}\left[1 - \delta_1 \le \tau_H^{tm} < 1, p^{tm}(1 - \tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r, \tilde{0}))\right] < \varepsilon. \tag{4.4}$$

Now let V^m be a Lévy process with Lévy density $\sigma = J - J^m$ and define $T^m := \inf\{t > 0 : V_t^m \neq 0\}$. Then V^m is a compound Poisson process and T^m is an exponential random variable with parameter m and independent of X (See [13]). Then we have

$$\begin{aligned}
&\mathbb{E}_{(r,\tilde{0})}\left[\tau_H^{tm} < 1-\delta_1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))\right] \\
&= \mathbb{E}_{(r,\tilde{0})}\left[T^{tm} > 1, \tau_H^{tm} < 1-\delta_1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))\right] \\
&+ \mathbb{E}_{(r,\tilde{0})}\left[T^{tm} \le 1, \tau_H^{tm} < 1-\delta_1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))\right].
\end{aligned}$$

Since $p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0})) \le c\frac{e^{mt}}{L^{d+\alpha}}$, we have

$$\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} \le 1, \tau_H^{tm} < 1-\delta_1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))\right] \le c\frac{e^{mt}}{L^{d+\alpha}}(1-e^{-mt}). \quad (4.5)$$

Similarly we also have

$$\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} \le 1, \tau_H^{0} < 1-\delta_1, p^{0}(1-\tau_H^{0}, X^{0}_{\tau_H}, (r,\tilde{0}))\right] \le c\frac{1}{L^{d+\alpha}}(1-e^{-mt}). \quad (4.6)$$

Take $t_1 > 0$ such that Eqs. 4.5 and 4.6 are less than ε for all $t \le t_1$. Next note that for $T^{tm} > 1$ and $\tau_H^{tm} < 1$, we have $\tau_H^{tm} = \tau_H^0$ and $X^{tm}_{\tau_H^{tm}} = X^0_{\tau_H^0}$. Hence it follows that

$$\begin{aligned}
&|\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} > 1, \tau_H^{tm} < 1-\delta_1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))\right] \\
&-\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} > 1, \tau_H^{0} < 1-\delta_1, p^{0}(1-\tau_H^{0}, X^{0}_{\tau_H^0}, (r,\tilde{0}))\right]| \\
&\le \mathbb{E}_{(r,\tilde{0})}\left[T^{tm} > 1, \tau_H^{0} < 1-\delta_1, |p^{tm}(1-\tau_H^{0}, X^{0}_{\tau_H^0}, (r,\tilde{0})) - p^{0}(1-\tau_H^{0}, X^{0}_{\tau_H^0}, (r,\tilde{0}))|\right] \\
&\le \sup_{s\in[\delta_1,1], x,y\in\mathbb{R}^d} |p^{tm}(s,x,y) - p^0(s,x,y)|. \qquad (4.7)
\end{aligned}$$

It follows from Lemma 4.3 that there exists $t_2 > 0$ such that $\sup_{s\in[\delta_1,1],x,y\in\mathbb{R}^d} |p^{tm}(s,x,y) - p^0(s,x,y)| < \varepsilon$ for $0 \le t \le t_2$. Now let $t_0 = t_1 \wedge t_2$. Then for any $0 \le t \le t_0$ we have from Eqs. 4.3-4.7

$$\begin{aligned}
&|r_H^{tm}(1,(r,\tilde{0}),(r,\tilde{0})) - r_H^0(1,(r,\tilde{0}),(r,\tilde{0}))| \\
&= |\mathbb{E}_{(r,\tilde{0})}[\tau_H^{tm} < 1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))] - \mathbb{E}_{(r,\tilde{0})}[\tau_H^{0} < 1, p^{0}(1-\tau_H^{0}, X^{0}_{\tau_H^0}, (r,\tilde{0}))]| \\
&\le |\mathbb{E}_{(r,\tilde{0})}[1 > \tau_H^{tm} > 1-\delta_1, \tau_H^{tm} < 1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))]| + \\
&|\mathbb{E}_{(r,\tilde{0})}[1 > \tau_H^{0} > 1-\delta_1, \tau_H^{0} < 1, p^{0}(1-\tau_H^{0}, X^{0}_{\tau_H^0}, (r,\tilde{0}))]| + \\
&|\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} \le 1, \tau_H^{tm} < 1-\delta_1, p^{tm}(1-\tau_H^{tm}, X^{tm}_{\tau_H^{tm}}, (r,\tilde{0}))\right]| + \\
&|\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} \le 1, \tau_H^{0} < 1-\delta_1, p^{0}(1-\tau_H^{0}, X^{0}_{\tau_H^0}, (r,\tilde{0}))\right]| \\
&+|\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} > 1, \tau_H^{0} < 1-\delta_1, p^{tm}(1-\tau_H^{0}, X^{0}_{\tau_H^0}, (r,\tilde{0}))\right] \\
&-\mathbb{E}_{(r,\tilde{0})}\left[T^{tm} > 1, \tau_H^{0} < 1-\delta_1, p^{0}(1-\tau_H^{0}, X^{0}_{\tau_H^0}, (r,\tilde{0}))\right]| \\
&< 5\varepsilon.
\end{aligned}$$

□

As in [3], we need to divide the Lipschitz open set D into a good set and a bad set. We recall several geometric facts about the Lipschitz open set.

Definition 4.5 Let $\varepsilon, r > 0$. We say that $G \subset \partial D$ is (ε, r)-good if for each point $p \in G$, the unit inner normal $\nu(p)$ exists and

$$B(p, r) \cap \partial D \subset \{x : |(x - p) \cdot \nu(p)| < \varepsilon |x - p|\}.$$

If G is an (ε, r)-good subset of ∂D, then using this definition we can construct a good subset $\mathcal{G}$ of the points near the boundary:

$$\mathcal{G} = \bigcup_{p \in G} \Gamma_r(p, \varepsilon),$$

where $\Gamma_r(p, \varepsilon) = \{x : (x - p) \cdot \nu(p) > \sqrt{1 - \varepsilon^2}|x - p|\} \cap B(p, r)$.

The next lemma is [3, Lemma 2.7] and it says the measure of the set of the bad points near the boundary is small. Note that even though [3, Lemma 2.7] is stated for a bounded Lipschitz domain, the proof remains true for a bounded Lipschitz open set.

Lemma 4.6 (Lemma 2.7 in [3]) *Suppose $\varepsilon \in (0, 1/2)$, $r > 0$ and that G is a measurable (ε, r)-good subset of ∂D. There exists $s_0(\partial D, G) > 0$ such that for all $s < s_0$*

$$|\{x \in D : \delta_D(x) < s\} \setminus \mathcal{G}| \le s \left[\mathcal{H}^{d-1}(\partial D \setminus G) + \varepsilon \left(3 + \mathcal{H}^{d-1}(\partial D)\right)\right].$$

The next lemma is about the existence of a good subset $G \subset \partial D$. Again the lemma remains true for a bounded Lipschitz open set D.

Lemma 4.7 (Lemma 2.8 in [3]) *For any $\varepsilon > 0$ there exists $r > 0$ such that an (ε, r)-good set $G \subset \partial D$ exists and*

$$\mathcal{H}^{d-1}(\partial D \setminus G) < \varepsilon.$$

The two lemmas above imply that

$$|\{x \in D : \delta_D(x) < s\} \setminus \mathcal{G}| \le s\varepsilon \left(4 + \mathcal{H}^{d-1}(\partial D)\right).$$

For any $\varepsilon \in (0, 1/4)$, we fix the (ε, r)-good set from Lemma 4.7 and construct $\mathcal{G}$ from G. We choose r to be smaller than the minimal distances between (finitely many) components of D. For any $x \in \mathcal{G}$, there exists $p(x) \in \partial D$ such that $x \in \Gamma_r(p(x), \varepsilon)$. Next we define inner and outer cones as follows

$$I_r\,(p(x)) = \{y : (y - p(x)) \cdot \nu\,(p(x)) > \varepsilon |y - p(x)|\} \cap B(p(x), r), \tag{4.8}$$

$$U_r\,(p(x)) = \{y : (y - p(x)) \cdot \nu\,(p(x)) < -\varepsilon |y - p(x)|\} \cap B(p(x), r). \tag{4.9}$$

It follows from [3, (2.20)] that there exists a half-space $H^*(x)$ such that

$$x \in H^*(x), \quad \delta_{H^*(x)}(x) = \delta_D(x), \quad I_r\,(p(x)) \subset H^*(x) \subset U_r\,(p(x))^c\,. \tag{4.10}$$

Now we are ready to prove Theorem 1.2.

Proof **of Theorem 1.2.** Fix $\varepsilon \in (0, 1/4)$, the (ε, r)-good set from Lemma 4.7 and the $\mathcal{G}$ constructed from G. From the definition of the trace we have

$$\begin{aligned} -t^{d/\alpha}\int_D r_D^m(t,x,x)dx &= t^{d/\alpha}\int_D \left(p_D^m(t,x,x) - p^m(t,x,x)\right)dx \\ &= t^{d/\alpha} Z_D^m(t) - t^{d/\alpha}\int_D p^m(t,x,x)dx \\ &= t^{d/\alpha} Z_D^m(t) - t^{d/\alpha}\int_D \left(p^0(t,x,x) - \left(p^0(t,x,x) - p^m(t,x,x)\right)\right)dx \\ &= t^{d/\alpha} Z_D^m(t) - C_1|D| + t^{d/\alpha}\int_D \left(p^0(t,x,x) - p^m(t,x,x)\right)dx. \end{aligned}$$

Hence it follows from Lemma 3.2 that in order to prove Theorem 1.2 we must show that for given $\varepsilon \in (0, 1/4)$ there exists a $t_0 > 0$ such that for any $0 < t < t_0$,

$$\left| t^{d/\alpha}\int_D r_D^m(t,x,x)dx - C_2\mathcal{H}^{d-1}(\partial D)t^{1/\alpha}\right| \le c(\varepsilon)t^{1/\alpha},$$

where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. As in the proof of [3, Theorem 1.1.] we split the region of integration into three sets

$$\begin{aligned} \mathcal{D}_1 &= \{x \in D \setminus \mathcal{G} : \delta_D(x) < s\}, \\ \mathcal{D}_2 &= \{x \in D \cap \mathcal{G} : \delta_D(x) < s\}, \\ \mathcal{D}_3 &= \{x \in D : \delta_D(x) \ge s\}, \end{aligned}$$

where s must be smaller than the s_0 given by Lemma 4.6. For small enough t we can take

$$s = t^{1/\alpha}/\sqrt{\varepsilon}.$$

It is shown in [3, (3.2) and (3.4)] that

$$t^{d/\alpha}\int_{\mathcal{D}_1\cup\mathcal{D}_3} r_D^0(t,x,x)dx \le c(\varepsilon)t^{1/\alpha} \tag{4.11}$$

where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence it follows from Lemma 2.1 and Eq. 4.11 that

$$t^{d/\alpha}\int_{\mathcal{D}_1\cup\mathcal{D}_3} r_D^m(t,x,x)dx \le c(\varepsilon)e^{2mt}t^{1/\alpha}. \tag{4.12}$$

Now we deal with the integral on $\mathcal{D}_2$. Let $H^*(x)$, $I_r(p(x))$, $U_r(p(x))$ be defined by Eqs. 4.8, 4.9 and 4.10. We have

$$I_r(p(x)) \subset H^*(x) \subset U_r(p(x))^c.$$

Since r is less than the minimal distances between components of D, we also have

$$I_r(p(x)) \subset D \subset U_r(p(x))^c.$$

Since $I_r(p(x)) \subset U_r(p(x))^c$, by an argument similar to that used in Lemma 3.4 we have

$$\begin{aligned} &\left| r_D^m(t,x,x) - r_{H^*(x)}^m(t,x,x)\right| \\ &\le r_{I_r(p(x))}^m(t,x,x) - r_{U_r(p(x))^c}^m(t,x,x) \\ &\le e^{2mt}\left(r_{I_r(p(x))}^0(t,x,x) - r_{U_r(p(x))^c}^0(t,x,x)\right). \end{aligned} \tag{4.13}$$

Now it follows from the argument after the statement of [3, Proposition 3.1.] and Eq. 4.13 that

$$t^{d/\alpha}\int_{\mathcal{D}_2}\left|r_D^m(t,x,x)-r_{H^*(x)}^m(t,x,x)\right|dx$$
$$\le ce^{2mt}\left(\varepsilon^{1-\alpha/2}\vee\sqrt{\varepsilon}\right)\mathcal{H}^{d-1}(\partial D)t^{1/\alpha}\int_0^\infty\left(r^{-d-\alpha+1}\wedge 1\right)dr.$$

Finally we will show that the integral

$$t^{d/\alpha}\int_{\mathcal{D}_2}r_{H^*(x)}^m(t,x,x)dx$$

gives the second term $C_2\mathcal{H}^{d-1}(\partial D)t^{1/\alpha}$ plus an error term of order $c(\varepsilon)t^{1/\alpha}$. Recall that

$$r_{H^*(x)}^m(t,x,x)=f_{H^*}^m(t,\delta_{H^*(x)})=f_H^m(t,\delta_D(x)).$$

Hence we have

$$t^{d/\alpha}\int_{\mathcal{D}_2}r_{H^*(x)}^m(t,x,x)dx$$
$$=t^{d/\alpha}\int_{\mathcal{D}_2}f_H^m(t,\delta_D(x))dx$$
$$=t^{d/\alpha}\int_D f_H^m(t,\delta_D(x))dx-t^{d/\alpha}\int_{\mathcal{D}_1\cup\mathcal{D}_3}f_H^m(t,\delta_D(x))dx.$$

By an argument similar to that used to get (4.12) we have that

$$t^{d/\alpha}\int_{\mathcal{D}_1\cup\mathcal{D}_3}f_H^m(t,\delta_D(x))dx\le c(\varepsilon)t^{1/\alpha},$$

where $c(\varepsilon)\to 0$ as $\varepsilon\to 0$. From the (approximate) scaling property of the relativistic stable process, we have

$$t^{d/\alpha}\int_D f_H^m(t,\delta_D(x))dx=\int_D f_H^{mt}(1,\delta_D(x)/t^{1/\alpha})dx.$$

Now apply Lemmas 4.2 and 4.4 to the function $r\to f_H^{mt}(1,r)$ and we get for small enough t

$$\left|\int_D f_H^{mt}(1,\delta_D(x)/t^{1/\alpha})dx-C_2\mathcal{H}^{d-1}(\partial D)t^{1/\alpha}\right|\le\varepsilon t^{1/\alpha}.$$

□

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