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Uniform Dimension Results for the Inverse Images of Symmetric Lévy Processes

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Abstract
We prove uniform Hausdorff and packing dimension results for the inverse images of a large class of real-valued symmetric Lévy processes. Our main result for the Hausdorff dimension extends that of Kaufman (C R Acad Sci Paris Sér I Math 300:281–282, 1985) for Brownian motion and that of Song et al. (Electron Commun Probab 23:10, 2018) for α-stable Lévy processes with $1 < \alpha < 2$. Along the way, we also prove an upper bound for the uniform modulus of continuity of the local times of these processes.

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1 Introduction

The inverse images of Lévy processes have been studied intensively. When $F = \{y\}$ is a single set, the question whether $\mathbb{P}(X^{-1}(F) \neq \emptyset) = 0$ boils down to whether y is polar for X, which is an important question in potential theory. As the sample paths of X are typically irregular, so the inverse images, in particular the level sets, are of a fractal nature. It is thus of interest to determine the fractal dimension of the set

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$X^{-1}(F)$, where F is a Borel subset of $\mathbb{R}$. The problems for determining the Hausdorff dimension and capacity of $X^{-1}(F)$ (when $F \subset \mathbb{R}$ is fixed) were studied by Hawkes [10,12] for strictly α-stable Lévy processes in $\mathbb{R}$, and by Khoshnevisan and Xiao [15] for general Lévy processes. Their methods are based on potential theory of Lévy processes. See Taylor [24] and Xiao [27] for further information on fractal properties of Lévy processes, where several interesting open problems have remained unsolved.

Motivated by the research in [10,12,15], an interesting question would be to provide a dimension formula for $X^{-1}(F)$ that holds simultaneously for all Borel sets $F \subset \mathbb{R}$. Such a stronger statement, if it is proved to hold, is customarily referred to as a uniform dimension result. The uniform dimension result has a wide applicability because it allows F to be random and dependent of the sample paths of X. For example, it allows us to compute $\dim_{\rm H} X^{-1}(F)$ when $F = X(E)$, where $E \subset [0, \infty)$ is a Borel set. More interestingly, the uniform dimension result is useful for the multifractal analysis of stochastic processes. For instance, [13,19,21] studied the multifractal structures of the local times at 0, denoted by $\{L(0,t), t \geq 0\}$, of a Lévy process $X = \{X(t), t \geq 0\}$ that hits points. See Sect. 4 for the definition and more information of local times of Lévy processes. An open problem is to investigate the multifractal structures of the local time processes $\{L(x,t), x \in \mathbb{R}\}$ (when t is fixed) or $\{L(x,t), t \geq 0, x \in \mathbb{R}\}$. For any result on the fractal dimension of the set F of points x where $L(x,t)$ has certain (fast or slow) oscillation behavior, one can use the uniform dimension result for the inverse images to derive the fractal dimension of the corresponding set of times $X^{-1}(F)$. This is worthy to pursue, but is beyond the scope of this paper.

The uniform dimension result for the level set $X^{-1}(x)$ is also fundamental in the geometric construction of the local times. Indeed, Barlow et al. [3] showed that a class of Lévy processes $\{X(t), t \geq 0\}$ with local times $\{L(x,t), x \in \mathbb{R}, t \geq 0\}$ satisfies

$$\mathbb{P}\left(L(x,t) = \mathcal{H}^{\phi}([0,t] \cap \{s : X(s) = x\}) \text{ for all } x \in \mathbb{R}, t \geq 0\right) = 1,$$

where ϕ is a sort of gauge function of the level sets. In other words, the local times of X can be obtained level-wise by computing certain Hausdorff measure of the level sets. Such a construction does not make sense if the uniform Hausdorff dimension result for the inverse images does not hold.

This paper is concerned with the uniform Hausdorff and packing dimension results for the inverse images of a symmetric Lévy process and continues the recent investigation of Song et al. [22], who proved a uniform Hausdorff dimension result for the inverse images of an α-stable Lévy process with $1 < \alpha < 2$, and the classical result of Kaufman [14] for Brownian motion. Let us recall their results.

Theorem 1.1 ([14,22]) *Let $X = \{X(t), t \geq 0\}$ be a strictly α-stable Lévy process with $1 < \alpha \leq 2$. For any $x \in \mathbb{R}$,*

$$\mathbb{P}^x\left(\dim_{\rm H} X^{-1}(F) = 1 - \frac{1}{\alpha} + \frac{\dim_{\rm H} F}{\alpha} \textit{ for all Borel sets } F \subseteq \mathbb{R}\right) = 1.$$

Our goal in the present paper is twofold. First, the proof of [22] relies on the exact scaling property satisfied by strictly stable Lévy processes. We intend to show that certain weak asymptotic behavior of the characteristic exponent ψ (see below) suffices to derive the uniform dimension result and such asymptotic behavior holds for many interesting examples. Second, we consider not only Hausdorff dimension but also packing dimension and thus provide two "dual" descriptions for the fractal behavior of the inverse images of Lévy processes.

Let $X = \{X(t), t \geq 0, \mathbb{P}^x\}$ be a real-valued Lévy process with characteristic (or Lévy) exponent ψ, that is, $\mathbb{E}^0[e^{i\lambda X(t)}] = e^{-t\psi(\lambda)}$. Throughout the paper, X is assumed to be symmetric, that is, X and $-X$ have the same distribution under $\mathbb{P}^0$. Consequently, ψ is real-valued and has the Lévy–Khintchine representation $\psi(\lambda) = \frac{1}{2}A\lambda^2 + 2\int_0^\infty (1-\cos(x\lambda))\,\nu(\mathrm{d}x)$, where $A \geq 0$ is the variance parameter of the Gaussian part of X and ν is called the Lévy measure satisfying $\nu(\{0\}) = 0$ and the integrability condition $\int_{\mathbb{R}\setminus\{0\}}(1 \wedge x^2)\nu(\mathrm{d}x) < \infty$. We suppose that X is a pure-jump Lévy process, i.e., $A = 0$.

Recall that a Borel measure μ on $\mathbb{R}$ is called unimodal with mode a if $\mu = c\delta_a + f(x)\mathrm{d}x$ with $c \geq 0$ and f increasing on $(-\infty, a)$, decreasing on (a, ∞). Unimodality is a time-dependent property for general Lévy processes. However, in the symmetric case, it is known that the distribution of $X(t)$ is unimodal for all $t > 0$ if and only if the Lévy measure ν of X is unimodal [25, p. 488]. In such case, it makes sense to say that X is a unimodal process. We refer to [20] for systematic accounts on Lévy processes.

In order to state our main result, we recall the weak scaling conditions introduced in [6].

Definition 1.2 We say that $\psi : [0,\infty) \to [0,\infty)$ satisfies the weak lower scaling condition at infinity if there exist constants $\underline{\alpha} \in \mathbb{R}$, $\underline{\theta} \geq 0$, and $\underline{c} \in (0, 1]$ such that

$$\psi(\lambda\theta) \geq \underline{c}\lambda^{\underline{\alpha}}\psi(\theta), \quad \lambda \geq 1, \theta > \underline{\theta},$$

and write $\psi \in \mathrm{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$. We say ψ satisfies the weak upper scaling condition at infinity if there exist $\overline{\alpha} \in \mathbb{R}$, $\overline{\theta} \geq 0$, and $\overline{C} \in [1, \infty)$ such that

$$\psi(\lambda\theta) \leq \overline{C}\lambda^{\overline{\alpha}}\psi(\theta), \quad \lambda \geq 1, \theta > \overline{\theta},$$

and write $\psi \in \mathrm{WUSC}(\overline{\alpha}, \overline{\theta}, \overline{C})$.

If $\underline{\theta} = 0$, we say that ψ satisfies the global weak lower scaling condition and the $\overline{\theta} = 0$ case is called the global weak upper scaling condition.

It is clear that $\underline{\alpha} \leq \overline{\alpha}$ in Definition 1.2. The following Theorems 1.3 and 1.4 are the main results of this paper. Theorem 1.3 extends [22, Th. 1.1], and Theorem 1.4 is new even for Brownian motion. We use $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{P}}$ to denote the Hausdorff dimension and the packing dimension of a set, respectively.

Theorem 1.3 (i) *Suppose that X is a unimodal symmetric pure-jump Lévy process in $\mathbb{R}$ with the Lévy exponent $\psi \in \mathrm{WLSC}(\alpha, 0, \underline{c}) \cap \mathrm{WUSC}(\alpha, \overline{\theta}, \overline{C})$ with $1 < \alpha < 2$ and some constants $\overline{\theta} > 0$, $\underline{c}$, and $\overline{C}$. Then, we have for all $x \in \mathbb{R}$*

$$\mathbb{P}^x\left(\dim_{\mathrm{H}} X^{-1}(F) \leq 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{H}} F}{\alpha} \text{ for all Borel sets } F \subseteq \mathbb{R}\right) = 1. \quad (1.1)$$

(ii) *Suppose that* $\psi \in \mathrm{WLSC}(\alpha, \underline{\theta}, \underline{c}) \cap \mathrm{WUSC}(\alpha, \overline{\theta}, \overline{C})$ *with* $\alpha \in (1, 2]$ *and* $\underline{\theta}, \overline{\theta} > 0$. *Then, we have*

$$\mathbb{P}^x \left(\dim_{\mathrm{H}} X^{-1}(F) \geq 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{H}} F}{\alpha} \text{ for all Borel sets } F \subseteq \mathbb{R} \right) = 1. \quad (1.2)$$

(iii) *Suppose that* $\psi \in \mathrm{WLSC}(\alpha, 0, \underline{c}) \cap \mathrm{WUSC}(\alpha, \overline{\theta}, \overline{C})$ *with* $\alpha \in (1, 2)$ *and* $\overline{\theta} > 0$. *Then, we have*

$$\mathbb{P}^x \left(\dim_{\mathrm{H}} X^{-1}(F) = 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{H}} F}{\alpha} \text{ for all Borel sets } F \subseteq \mathbb{R} \right) = 1. \quad (1.3)$$

Theorem 1.4 (i) *Suppose that the condition of the first part of Theorem 1.3 holds. Then, we have for all* $x \in \mathbb{R}$

$$\mathbb{P}^x \left(\dim_{\mathrm{P}} X^{-1}(F) \leq 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{P}} F}{\alpha} \text{ for all Borel sets } F \subseteq \mathbb{R} \right) = 1. \quad (1.4)$$

(ii) *Suppose that the condition of Part* (ii) *of Theorem 1.3 holds. Then, we have for all* $x \in \mathbb{R}$

$$\mathbb{P}^x \Bigg(\dim_{\mathrm{P}} X^{-1}(F) = 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{P}} F}{\alpha} \text{ for all Borel sets } F \subseteq \mathbb{R} \text{ with } \dim_{\mathrm{P}} F = \dim_{\mathrm{H}} F \Bigg) = 1. \quad (1.5)$$

Remark 1.5 1. The condition that $\dim_{\mathrm{P}} F = \dim_{\mathrm{H}} F$ in Theorem 1.4 is a regularity condition on the set F and is technical in nature. Even though such condition is satisfied by many fractal sets, it is natural to ask whether one may remove it. In the case of Brownian motion, an affirmative answer can be proved by applying its uniform modulus of continuity and the asymptotic property of its local times. However, in the general Lévy processes, we have not been able to do so due to a difficulty caused by the jumps of X.

2. Item (ii) of Theorem 1.4 follows from item (ii) of Theorem 1.3. Indeed, $\dim_{\mathrm{P}} X^{-1}(F) \geq \dim_{\mathrm{H}} X^{-1}(F) \geq 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{H}} F}{\alpha} = 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{P}} F}{\alpha}$ under the regularity condition on F.
3. As explained in [22] when $0 < \alpha < 1$, the uniform Hausdorff dimension estimate for the inverse images for symmetric stable processes cannot be true. Therefore, one cannot expect Theorem 1.3 to hold when $0 < \alpha < 1$. The case for $\alpha = 1$ is still open even for the Cauchy process.
4. It is an interesting question to study the dimensions of $X^{-1}(F)$ when the upper and lower scaling indices of ψ are different (i.e., $\underline{\alpha} < \overline{\alpha}$). In this case, it is possible to extend Lemma 3.1 and Theorem 4.4 so that one can derive upper and lower bounds for the Hausdorff and packing dimensions of $X^{-1}(F)$ in terms of $\underline{\alpha}$, $\overline{\alpha}$, and the dimensions of F. However, since F may vary arbitrarily, there is no hope

to obtain equalities in general. Hence, we have chosen to state Theorems 1.3 and 1.4 to give explicit formulae for $\dim_{\mathrm{H}} X^{-1}(F)$ and $\dim_{\mathrm{P}} X^{-1}(F)$.

5. Non-uniform dimension results on the inverse images of Lévy-type processes have been obtained in [17], and it would be interesting to obtain uniform dimension results for these processes.

The general strategy for proving Theorems 1.3 and 1.4 is similar to that of [22]. To show the uniform (in set F) upper bounds in (1.1) and (1.4), we will prove a covering principle for the inverse images $X^{-1}(F)$ by applying the recent contribution of [9] on the hitting times of a class of Lévy processes. On the other hand, for proving the lower bound in (1.2), we investigate regularity properties of the local time $L(x,\cdot)$ of X, which can be extended to a Borel measure supported by the level sets of X. This allows us to construct a family of random measures carried by the inverse image $X^{-1}(F)$ and to establish the desired uniform lower bound by using Frostman's lemma.

This paper is organized as follows. We present preliminary material in Sect. 2. Theorems 1.3 and 1.4 are proved in Sects. 3 and 4, respectively. Some examples are given in Sect. 5. Throughout the paper, for $f, g : \mathbb{R} \to \mathbb{R}$, we write $f \asymp g$ if the ratio of the two functions is bounded from above and from below by some positive finite constants. Universal constants are denoted by c, C which may differ from line to line. Specific constants are denoted by c_1, c_2, K_1, K_2, etc. Denote by $\mathbb{E}^x$ the expectation with respect to $\mathbb{P}^x$ and for simplicity, write $\mathbb{P} = \mathbb{P}^0$ and $\mathbb{E} = \mathbb{E}^0$.

2 Preliminaries

2.1 Fractal Dimensions

For definition of Hausdorff dimension, we refer to the monograph of Falconer [7]. In the following, we recall from [7] the definition of packing measure and dimension.

Let $s > 0$ be a constant. For any $F \subseteq \mathbb{R}^d$, the *s-dimensional packing measure* of F is defined by

$$\mathcal{P}^s(F) = \inf\left\{ \sum_{i=1}^{\infty} \mathcal{P}_0^s(F_i) : \ F \subset \bigcup_{i=1}^{\infty} F_i \right\},$$

where $\mathcal{P}_0^s(F) =\downarrow \lim_{\delta\to 0} \mathcal{P}_\delta^s(F)$ and

$$\mathcal{P}_\delta^s(F) = \sup\left\{ \sum_{i=1}^{\infty} (2r_i)^s \right\},$$

where the supremum is taken over all collections $\{B_i\}$ of disjoint balls of radii r_i at most δ with centers in F. The packing dimension of F is defined as

$$\dim_{\mathrm{P}} F = \sup\{s \geq 0 : \mathcal{P}^s(F) = \infty\} = \inf\{s \geq 0 : \mathcal{P}^s(F) = 0\}.$$

It can be verified that the packing dimension is stable under countable union in the sense that

$$\dim_{\mathrm{P}}\left(\bigcup_{i=1}^{\infty} F_i\right) = \sup_i \dim_{\mathrm{P}} F_i. \tag{2.1}$$

For any bounded set $F \subset \mathbb{R}^d$, let $N_\delta(F)$ be the smallest number of sets of diameter at most δ that covers F. Then, the upper box dimension of F is defined by

$$\overline{\dim}_{\mathrm{B}} F = \limsup_{\delta\to 0} \frac{\ln N_\delta(F)}{-\ln\delta}.$$

Note that for any set $F \subset \mathbb{R}^d$ (see [7, Equation (3.27)])

$$\dim_{\mathrm{H}} F \le \dim_{\mathrm{P}} F \le \overline{\dim}_{\mathrm{B}} F. \tag{2.2}$$

Moreover, packing and upper box dimensions are related by the following regularization procedure (cf. [7]):

$$\dim_{\mathrm{P}} F = \inf\left\{\sup_n \overline{\dim}_{\mathrm{B}} F_n \,:\, F \subset \bigcup_{i=1}^{\infty} F_n\right\}, \tag{2.3}$$

where the infimum is taken over all $\{F_n\}$ such that $F \subset \bigcup_{i=1}^{\infty} F_n$.

2.2 Weak Scaling Condition

Now, we focus on the weak scaling conditions. We start with some notation. Let $\phi : [0,\infty) \to [0,\infty)$ be a continuous and nondecreasing function. We define its generalized inverse as

$$\phi^{-1}(u) = \inf\{s \ge 0 : \phi(s) \ge u\}, \quad u \in [0, \phi(\infty)].$$

Note that ϕ^{-1} is nondecreasing and it is left continuous and has a right limit at every point. Also, it is easy to see $\phi^{-1}(\phi(u)) \le u$ and $\phi(\phi^{-1}(u)) = u$.

If ϕ is not necessarily nondecreasing, we define the maximal function ϕ^* as

$$\phi^*(y) = \sup\{\phi(x) : 0 \le x \le y\}.$$

Motivated by the relation $\inf\{s : \phi(s) \ge u\} = \inf\{s : \phi^*(s) \ge u\}$, we define $\phi^{-1} := (\phi^*)^{-1}$. Note that we still have $\phi^{-1}(\phi(u)) \le u$ and $\phi(\phi^{-1})(u) = u$.

Let us review some consequences of the weak scaling conditions. Each point is used in a specific stage of the proof.

- The global weak lower scaling condition implies an asymptotic result for the hitting times of X on a compact interval, see Theorem 2.1 below.

- We need in the proof that a re-scaled version of X, denoted by $Y_b = \{bX(b^{-\alpha}t), t \geq 0\}$ for the moment, satisfies $\mathbb{P}^0(Y_b(1) \in [2,3]) > c$ uniformly for all large b. This property is guaranteed whenever the characteristic exponent of X satisfies $\psi \in \mathrm{WLSC}(\alpha, \theta, \underline{c}) \cap \mathrm{WUSC}(\alpha, \theta, \overline{C})$ with $0 < \alpha < 2$ and that X is unimodal, as suggested by Theorem 2.2 below.

Denote by $T_A = \inf\{t > 0 : X(t) \in A\}$ the first hitting time of $A \subset \mathbb{R}$ by X. The following theorem is taken from [9] which provides a sharp tail probability estimate for the first hitting time of an interval.

Theorem 2.1 ([9, Th. 5.5]) *Suppose that* $\psi \in \mathrm{WLSC}(\underline{\alpha}, 0, \underline{c})$ *with* $\underline{\alpha} > 1$. *Then, for any* $x \in \mathbb{R}$ *with* $|x| > R$,

$$\mathbb{P}^x(T_{[-R,R]} > t) \asymp \frac{V(|x|-R)K(|x|)}{V(|x|)t\psi^{-1}(1/t)} \wedge 1, \quad t > 1/\psi^*(1/R),$$

where the comparability constant depends only on the scaling characteristics,

$$K(x) = \frac{1}{\pi}\int_0^\infty (1-\cos xs)\frac{1}{\psi(s)}\mathrm{d}s,$$

and $V(x)$, $x \geq 0$, *is the potential measure of the interval* $[0, x]$ *which satisfies*

$$V(r) \asymp \frac{1}{\sqrt{\psi^*(1/r)}}.$$

Recall the following result on the lower bound for the transition density of unimodal symmetric Lévy processes with the weak scaling properties.

Theorem 2.2 ([6, Th. 21]) *Let* X *be a unimodal symmetric Lévy process in* $\mathbb{R}$ *with characteristic exponent* ψ. *If* $\psi \in \mathrm{WLSC}(\underline{\alpha}, \theta, \underline{c}) \cap \mathrm{WUSC}(\overline{\alpha}, \theta, \overline{C})$ *with* $0 < \underline{\alpha}, \overline{\alpha} < 2$, *then there exist constants* c^* *and* r_0 *such that the transition density* $p(t,x)$ *satisfies*

$$p(t,x) \geq c^*\left(\psi^{-1}(1/t) \wedge \frac{t\psi^*(1/|x|)}{|x|}\right) \quad \text{if } t > 0,\ t\psi^*(\theta/r_0) < 1,\ \text{and}\ |x| < r_0/\theta.$$

3 Proof for the Upper Bounds

Let us start with the upper bound. We establish a covering principle for the inverse images of X, which is a reminiscence of [22, Lemma 2.2]. Let $\mathcal{U}_n$ be any partition of $\mathbb{R}$ with intervals of length 2^{-n} and $\mathcal{D}_n(\overline{\alpha})$ be any partition of $[0, \infty)$ with length $2^{-n\overline{\alpha}}$.

Lemma 3.1 *Suppose that* $\psi \in \mathrm{WLSC}(\alpha, 0, \underline{c}) \cap \mathrm{WUSC}(\alpha, \theta, \overline{C})$ *with* $1 < \alpha < 2$ *and for some constant* $\theta > 0$. *Let* $\delta > \alpha - 1$ *and* $T > 0$. $\mathbb{P}^x$*-a.s. for all integer* n *sufficiently large and each* $U \in \mathcal{U}_n$, $X^{-1}(U) \cap [0, T)$ *can be covered by* $2 \cdot 2^{n\delta}$ *intervals from* $\mathcal{D}_n(\alpha)$.

Proof *Step* 1 For a fixed $U \in \mathcal{U}_n$, write $U = (z - \frac{2^{-n}}{2}, z + \frac{2^{-n}}{2})$ for some $z \in \mathbb{R}$. Define $\tau_0 = 0$ and for all integer $k \geq 1$,

$$\tau_k = \inf\{s > \tau_{k-1} + 2^{-n\alpha} : X(s) \in U\}$$

with the convention that $\inf \emptyset = \infty$. From the fact that $X^{-1}(U) \subset \bigcup_{i=0}^{\infty}[\tau_i, \tau_i + 2^{-n\alpha}]$, we note that for any $T > 0$,

$$\{X^{-1}(U) \cap [0, T) \text{ cannot be covered by } k \text{ intervals of length } 2^{-n\alpha}\} \subset \{\tau_k < T\}.$$

Due to the right continuity of the sample paths of X, we observe that $X(\tau_{k-1})$ belongs to the closure of U as $\tau_{k-1} < T$. By the strong Markov property, we obtain

$$\begin{aligned}\mathbb{P}^x(\tau_k < T) &= \mathbb{P}^x(\tau_k < T | \tau_{k-1} < T)\mathbb{P}^x(\tau_{k-1} < T) \\ &\leq \sup_{y \in \overline{U}} \mathbb{P}^y \left(\inf_{2^{-n\alpha} \leq t \leq T} |X(s) - z| \leq \frac{2^{-n}}{2} \right) \mathbb{P}^x(\tau_{k-1} < T) \\ &\leq \sup_{y \in \overline{U}} \mathbb{P}^y \left(\inf_{2^{-n\alpha} \leq t \leq T} |X(s) - y| \leq 2^{-n} \right) \mathbb{P}^x(\tau_{k-1} < T).\end{aligned}$$

Define a sequence of intermediate processes $Y_n = \{2^n X(2^{-n\alpha}t), t \geq 0\}$. It follows from the spatial homogeneity of Lévy processes that for any $y \in \mathbb{R}$,

$$\mathbb{P}^y \left(\inf_{2^{-n\alpha} \leq t \leq T} |X(s) - y| \leq 2^{-n} \right) = \mathbb{P} \left(\inf_{1 \leq t \leq T 2^{n\alpha}} |Y_n(t)| \leq 1 \right) := p_n.$$

By induction, we obtain

$$\mathbb{P}^x(\tau_k < T) \leq p_n^k.$$

Step 2 Now, we intend to prove that $1 - p_n \geq C_T 2^{-n(\alpha-1)}$, thus find an upper bound for p_n. Let T_A^n be the hitting time of Y_n to any interval A. Considering the complement of the event associated with p_n, we have by the independence of increments that

$$\begin{aligned}1 - p_n &\geq \mathbb{P}\Big(2 \leq |Y_n(1)| \leq 3, \ \inf\{t \geq 1 : Y_n(t) - Y_n(1) \in [-4, -1]\} \geq T 2^{n\alpha} + 1\Big) \\ &= \mathbb{P}\big(2 \leq |Y_n(1)| \leq 3\big)\, \mathbb{P}\big(T^n_{[-4,-1]} > T 2^{n\alpha}\big).\end{aligned}$$

We proceed by looking for lower bounds for these events on Y_n. First,

$$\mathbb{P}(2 \leq |Y_n(1)| \leq 3) = \mathbb{P}(2 \cdot 2^{-n} \leq |X(2^{-n\alpha})| \leq 3 \cdot 2^{-n}) \geq \int_{2 \cdot 2^{-n}}^{3 \cdot 2^{-n}} p(2^{-n\alpha}, x) \mathrm{d}x.$$

For n sufficiently large, $2^{-n\alpha}\psi^*(\theta/r_0) < 1$ and $3 \cdot 2^{-n} < r_0/\theta$. Since $\psi \in \mathrm{WLSC}(\alpha, 0, \underline{c}) \cap \mathrm{WUSC}(\alpha, \theta, \overline{C})$ with $1 < \alpha < 2$, it follows from [6, Remark

4] we have $\psi^{-1} \in \mathrm{WUSC}(1/\alpha, 0, \underline{c}^{-1/\alpha}) \cap \mathrm{WLSC}(1/\alpha, \psi(\overline{\theta}), \overline{C}^{-1/\alpha})$. Note that for $x \in [2\cdot 2^{-n}, 3\cdot 2^{-n}]$ and $t = 2^{-n\alpha}$, we have $\psi^{-1}(2^{n\alpha}) \asymp 2^n$. Since $\psi \in \mathrm{WLSC}(\alpha, 0, \underline{c}) \cap \mathrm{WUSC}(\alpha, \theta, \overline{C})$, together with [6, Proposition 2], we have $\frac{t\psi^*(1/|x|)}{|x|} \asymp 2^{-n(\alpha-1)}\psi^*(2^n) \asymp 2^n$. This shows that $\psi^{-1}(1/t)$ is comparable to $\frac{t\psi^*(1/|x|)}{|x|}$ for $t = 2^{-n\alpha}$ and $x \in [2\cdot 2^{-n}, 3\cdot 2^{-n}]$. It follows from Theorem 2.2 and the weak lower scaling property of ψ that $p(2^{-n\alpha}, x) \geq \frac{2^{-n\alpha}\psi^*(1/x)}{x} \geq c2^n$ on $x \in [2\cdot 2^{-n}, 3\cdot 2^{-n}]$. Therefore,

$$\mathbb{P}(2 \leq |Y_n(1)| \leq 3) \geq c > 0$$

uniformly for all sufficiently large n.

Second, observe that Y_n under $\mathbb{P}$ has the characteristic exponent $\psi_n(\lambda) = 2^{-n\alpha}\psi(2^n\lambda)$, and it is easy to check that $\psi_n \in \mathrm{WLSC}(\alpha, 0, \underline{c})$ with the same scaling characteristics as those of ψ. Hence, by applying the spatial homogeneity of X and Theorem 2.1 with $x = \frac{5}{2}$, $R = \frac{3}{2}$, $t = T2^{n\alpha}$, we arrive at

$$\mathbb{P}\Big(T^n_{[-4,-1]} > T2^{n\overline{\alpha}}\Big) = \mathbb{P}^{5/2}\left(T^n_{[-3/2,3/2]} > T2^{n\alpha}\right) \asymp \frac{1}{T2^{n\alpha}\psi_n^{-1}(T^{-1}2^{-n\alpha})}.$$

Note that if $f(x) = ah(bx)$, then $f^{-1}(x) = b^{-1}h^{-1}(a^{-1}x)$. Applying this relation to $f = \psi_n$, $h = \psi$, $a = 2^{-n\alpha}$, $b = 2^n$, we obtain $\psi_n^{-1}(T^{-1}2^{-n\alpha}) = 2^{-n}\psi^{-1}(2^{n\alpha}\cdot T^{-1}2^{-n\alpha}) = 2^{-n}\psi^{-1}(T^{-1})$. Therefore,

$$\mathbb{P}\Big(T^n_{[-4,-1]} > T2^{n\alpha}\Big) \asymp C_T 2^{-n(\alpha-1)},$$

as desired.

Step 3 Define the event $A_n^\delta(T)$ by

$$A_n^\delta(T) = \Big\{\exists U \in \mathcal{U}_n \cap [-K, K] \text{ s.t. } X^{-1}(U) \cap [0, T] \text{ cannot be covered by } 2^{n\delta} \text{ intervals of length } 2^{-n\alpha}\Big\},$$

where $U \in \mathcal{U}_n \cap [-K, K]$ means that $U \in \mathcal{U}_n$ and $U \subset [-K, K]$. We have for $\delta > \alpha - 1$,

$$\sum_{n=1}^{\infty}\mathbb{P}^x(A_n^\delta(T)) \leq \sum_{n=1}^{\infty} 2K2^n(p_n)^{2^{n\delta}} \leq 2K\sum_{n=1}^{\infty}2^n\big(1 - c_T 2^{-n(\alpha-1)}\big)^{2^{n\delta}}$$
$$= 2K\sum_{n=1}^{\infty}\exp\left(n(\ln 2) - C_T 2^{n(\delta-\alpha+1)}\right) < \infty.$$

The conclusion for all $U \subset [-K, K] \cap \mathcal{U}_n$ follows from the Borel–Cantelli Lemma. Letting $K \to \infty$ completes the proof. □

Proof of Theorem 1.3 (i) Suppose that $\psi \in \mathrm{WLSC}(\alpha, 0, \underline{c}) \cap \mathrm{WUSC}(\alpha, \theta, \overline{C})$). The proof is similar to the proof of [22, Theorem 1.1], and we provide the details for the reader's convenience. Let F be any Borel set and take $\gamma > \dim_{\mathrm{H}} F$ and $\delta > \alpha - 1$. There exists a sequence of intervals $\{U_i\}$ of length 2^{-n_i} such that $F \subset \cup_{i=1}^{\infty} U_i$ and $\sum_{i=1}^{\infty} 2^{-n_i \gamma} < 1$. For any $T > 0$ it follows from Lemma 3.1, we have

$$X^{-1}(F) \cap [0, T] \subset \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{2 \cdot 2^{n_i \delta}} I_{i,k},$$

where $I_{i,k}$ are in $\mathcal{D}_{n_i}(\alpha)$. This implies

$$\sum_{i=1}^{\infty} \sum_{k=1}^{2 \cdot 2^{n_i \delta}} \mathrm{diam}(I_{i,k})^{\frac{\gamma+\delta}{\alpha}} = 2 \cdot 2^{n_i \delta} \sum_{i=1}^{\infty} (2^{-n_i \alpha})^{\frac{\gamma+\delta}{\alpha}} = 2 \sum_{i=1}^{\infty} 2^{-n_i \gamma} < \infty.$$

Hence,

$$\dim_{\mathrm{H}} \left(X^{-1}(F) \cap [0, T] \right) \leq \frac{\gamma + \delta}{\alpha}.$$

Letting $\gamma \downarrow \dim_{\mathrm{H}} F$, $\delta \downarrow \alpha - 1$, and $T \uparrow \infty$ (all along rational numbers) gives

$$\dim_{\mathrm{H}} X^{-1}(F) \leq \frac{\dim_{\mathrm{H}} F + \alpha - 1}{\alpha}.$$

□

Now, we establish the upper bound for the packing dimension. By the second point of Remark 1.5, it suffices to prove the upper bound of Theorem 1.4.

Proof of Theorem 1.4 (i) We will first prove that for any given $T > 0$,

$$\mathbb{P}^x \left(\overline{\dim}_{\mathrm{B}} (X^{-1}(F) \cap [0, T]) \leq 1 - \frac{1}{\alpha} + \frac{\overline{\dim}_{\mathrm{B}}(F)}{\alpha} \right) = 1. \tag{3.1}$$

Let $\theta > \overline{\dim}_{\mathrm{B}}(F)$ so that

$$N_{2^{-n}}(F) 2^{-n\theta} \to 0 \text{ as } n \to \infty.$$

Then, there exist intervals $\{U_i\}$ with length 2^{-n} such that $F \subset \cup_{i=1}^{N_{2^{-n}}(F)} U_i$. Let $\delta > \alpha - 1$. It follows from Lemma 3.1 that for all n sufficiently large,

$$X^{-1}(F) \cap [0, T] \subseteq \bigcup_{i=1}^{N_{2^{-n}}(F)} \left(X^{-1}(U_i) \cap [0, T] \right) \subseteq \bigcup_{i=1}^{N_{2^{-n}}(F)} \bigcup_{k=1}^{2 \cdot 2^{n\delta}} V_{ik},$$

where $|V_{ik}| = 2^{-n\alpha}$. Hence,

$$N_{2^{-n\alpha}}(X^{-1}(F) \cap [0, T]) \le 2^{1+n\delta} N_{2^{-n}}(F).$$

Let $d = \frac{\theta+\delta}{\alpha}$. Then, we have

$$N_{2^{-n\alpha}}(X^{-1}(F) \cap [0, T])(2^{-n\alpha})^d \le 2N_{2^{-n}}(F)2^{-n\theta} \to 0 \text{ as } n \to \infty$$

and this implies that $\overline{\dim}_B X^{-1}(F) \cap [0, T] \le d$ a.s. Letting $\theta \downarrow \overline{\dim}_B(F)$ and $\delta \downarrow \alpha - 1$ proves (3.1).

Now, take any cover of $F \subset \bigcup F_n$. It follows from (2.1), (2.2), and (3.1) that

$$\begin{aligned} \dim_P\left(X^{-1}(F) \cap [0, T]\right) &\le \dim_P \bigcup_{n=1}^{\infty} \left(X^{-1}(F_n) \cap [0, T]\right) \\ &= \sup_n \dim_P\left(X^{-1}(F_n) \cap [0, T]\right) \\ &\le \sup_n \overline{\dim}_B\left(X^{-1}(F_n) \cap [0, T]\right) \le 1 - \frac{1}{\alpha} + \frac{\overline{\dim}_B F_n}{\alpha}. \end{aligned} \tag{3.2}$$

Since the left hand side of (3.2) does not depend on $\{F_n\}$, an application of (2.3) yields

$$\mathbb{P}^x\left(\dim_P\left(X^{-1}(F) \cap [0, T]\right) \le 1 - \frac{1}{\alpha} + \frac{\dim_P F}{\alpha}\right) = 1.$$

Finally, we let $T \to \infty$ and this proves (1.4). □

4 Proof for the Lower Bound

We move to prove the lower bound in (1.2). We first establish a uniform Hölder-type condition for the local times of such processes by using the method of moments which is similar to Khoshnevisan et al. [16] or Xiao [26].

We first recall the *lower index* β^{low} of an arbitrary Lévy process introduced by Blumenthal and Getoor [4],

$$\beta^{\text{low}} = \sup\left\{\gamma \ge 0 : \lim_{|\xi|\to\infty} \|\xi\|^{-\gamma} \text{Re}\psi(\xi) = \infty\right\},$$

where $\text{Re}\psi(\xi)$ represents the real part of its argument. Since the process X is symmetric, we have $\text{Re}\psi(\xi) = \psi(\xi)$. Also, notice that when $\psi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$ we have $\beta^{\text{low}} \ge \underline{\alpha}$.

Let $A \subset [0, \infty)$ be a Borel set and define the occupation measure $\mu_A(\cdot)$ by

$$\mu_A(\cdot) = m\big(\{t \in A : X(t) \in \cdot\}\big),$$

where $m(\cdot)$ is the Lebesgue measure in $\mathbb{R}$. For any Borel set $A \subset [0, \infty)$, if $\mu_A \ll m$, then we define a local time of X on A by

$$L(x, A) := \frac{\mathrm{d}\mu_A}{\mathrm{d}m}(x).$$

In this paper, we only consider $A = [0, \infty)$ and, in this case, Hawkes [11] showed that a necessary and sufficient condition for the existence of local times of a Lévy process X with exponent ψ is $\mathrm{Re}\big(\frac{1}{1+\psi(\xi)}\big) \in L^1(\mathbb{R})$. We will write $L(x, t)$ for $L(x, [0, t])$. If there is a modification of the local time such that it is continuous in (x, t), we say that X has a jointly continuous local time. Necessary and sufficient conditions for the joint continuity of the local times of Lévy processes have been proved by Barlow and Hawkes [2], Barlow [1], and by Marcus and Rosen [18] using different methods.

Since $\psi \in \mathrm{WLSC}(\underline{\alpha}, 0, \underline{c})$, it follows from [16, Theorem 2.1] that there exists a square integrable local time for X. We also note that under the current setting (with $N = 1$) the proof of [16, Theorem 3.2] holds true. Consequently, [16, Equations (3.16) and (3.17)] hold true as well. This establishes the following estimates for the local time of X.

Lemma 4.1 [16, Lemma 4.2] *Suppose that X is a symmetric unimodal Lévy process and $\psi \in \mathrm{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$ with $\underline{\alpha} > 1$. For any $\gamma \in (0, \frac{\underline{\alpha}-1}{2})$, there exist constants $b_1, b_2 > 0$ and $0 < K_1, K_2 < \infty$ such that for any interval $I = [a, a+h]$, $a \geq 0$, $x, y \in \mathbb{R}$, and $u > 0$,*

$$\mathbb{P}\left(L(X_t + x, I) \geq h^{1-\frac{1}{\underline{\alpha}}} u^{\frac{1}{\underline{\alpha}}}\right) \leq K_1 e^{-b_1 u},$$

and

$$\mathbb{P}\left(|L(X_t + x, I) - L(X_t + y, I)| \geq h^{1-\frac{1+\gamma}{\underline{\alpha}}} |x - y|^{\gamma} u^{\frac{1+\gamma}{\underline{\alpha}}}\right) \leq K_2 e^{-b_2 u},$$

where either $t = 0$ or a.

We need to estimate the local oscillation of the process X, which is given in the following lemma. It is a direct consequence of a general result for Lévy-type processes.

Lemma 4.2 [5, Th. 5.1] *Assume that X is a symmetric Lévy process. Then, there exists a constant c such that for any $t,\ r > 0$,*

$$\mathbb{P}\Big(\sup_{0 \leq s \leq t} |X(s)| > r\Big) \leq ct\psi^*(1/r).$$

If $\psi \in \mathrm{WUSC}(\delta, \overline{\theta}, \overline{C})$, we have a versatile version of Lemma 4.2.

Lemma 4.3 *Suppose that $\psi \in \mathrm{WUSC}(\delta, \overline{\theta}, \overline{C})$ with $\delta \in (0, 2]$. Then, for $r \leq \frac{1}{\overline{\theta}}$, we have*

$$\mathbb{P}\Big(\sup_{0 \leq s \leq t} |X(s)| > r\Big) \leq ctr^{-\delta}.$$

Proof Applying [8, Proposition 1] to our process X, we have

$$\psi^*(|x|) \leq 12\psi(x), \quad x \in \mathbb{R}. \tag{4.1}$$

Hence, it follows from Lemma 4.2, the weak upper scaling condition, and (4.1) that for $r \leq \frac{1}{\overline{\theta}}$

$$\mathbb{P}\left(\sup_{s\leq t}|X(s)| > r\right) \leq \overline{C}t\psi^*(1/r) \leq 12\overline{C}t\psi(1/r) \leq c\,t\psi(\overline{\theta})\overline{\theta}^{-\delta}r^{-\delta}.$$

This proves lemma. □

Now, we are ready to prove a uniform Hölder condition for the local times for X. The proof is similar to that of [16, Theorem 4.3] or [26, Theorems 1.1 and 1.2] with obvious modifications. We provide the details for the reader's convenience. For an interval $I \subset [0, \infty)$, we define $L^*(I) = \sup_{x\in\mathbb{R}} L(x, I)$ to be the maximum local time of X on I.

Theorem 4.4 *Suppose that X is a symmetric unimodal Lévy process in $\mathbb{R}$ and its characteristic exponent ψ satisfies*

$$\psi \in \mathrm{WLSC}(\alpha, \underline{\theta}, \underline{c}) \cap \mathrm{WUSC}(\alpha, \overline{\theta}, \overline{C})$$

with $\alpha > 1$ and for some constants $\underline{c}, \overline{C} > 0$. Let L be its jointly continuous local time and fix $\tau > 0$ and $N > 0$. Then, we have

$$\limsup_{r\to 0} \frac{L^*([\tau - r, \tau + r])}{r^{1-\frac{1}{\alpha}}(\ln\ln r^{-1})^{\frac{1}{\alpha}}} < \infty, \tag{4.2}$$

and

$$\limsup_{r\to 0} \sup_{I\subset[0,N], m(I)<r} \frac{L^*(I)}{r^{1-\frac{1}{\alpha}}\ln(r^{-1})^{\frac{1}{\alpha}}} < \infty. \tag{4.3}$$

Proof We first prove (4.2). Let $g(r) = r^{1-\frac{1}{\alpha}}(\ln\ln\frac{1}{r})^{\frac{1}{\alpha}}$ and $C_n = [s, s + \frac{1}{2^n}]$ for any $s \geq 0$. We will prove that

$$\limsup_{n\to\infty} \frac{L^*(C_n)}{g(2^{-n})} < \infty. \tag{4.4}$$

Since $2^{-n/\alpha}n^\beta < \theta$ for all but finitely many n's, it follows from Lemma 4.3 and the fact $\psi \in \mathrm{WUSC}(\alpha, \theta, \overline{C})$ that for all sufficiently large n,

$$\mathbb{P}\left(\sup_{t\in C_n}|X_t - X_s| > 2^{-n/\alpha}n^\beta\right) \leq 2^{-n}(2^{-n/\alpha}n^\beta)^\alpha = n^{-\alpha\beta}.$$

We choose $\beta > \frac{1}{\alpha}$. Then, the Borel–Cantelli lemma gives that a.s.

$$\sup_{t \in C_n} |X_t - X_s| \leq 2^{-n/\alpha} n^\beta$$

for all but finitely many n's.

Now, let $\theta_n := 2^{-n/\alpha} (\ln n)^{-1/\alpha}$. Define

$$G_n = \{x : |x| \leq 2^{-n/\alpha} n^\beta, \quad x = p\theta_n \text{ for some } p \in \mathbb{Z}\}.$$

Note that the number of elements in G_n is at most $\frac{2^{-n/\alpha} n^\beta}{\theta_n} = n^\beta (\ln n)^{1/\alpha}$. Choose a constant a_1 so that $b_1 a_1 - \beta > 1$, where b_1 is a constant from Lemma 4.1. Then, Lemma 4.1 implies

$$\begin{aligned}
&\mathbb{P}\left(\max_{x \in G_n} L(X_s + x, C_n) \geq a_1^{1/\alpha} g(2^{-n})\right) \\
&= \mathbb{P}\left(\max_{x \in G_n} L(X_s + x, C_n) \geq (2^{-n})^{1-\frac{1}{\alpha}} (a_1 \ln\ln 2^n)^{1/\alpha}\right) \\
&= (\text{\# of elements in } G_n) \times K_1 e^{-b_1 a_1 \ln\ln 2^n} \\
&\leq n^\beta (\ln n)^{1/\alpha} \times K_1 e^{-b_1 a_1 \ln\ln 2^n} = K_1 e^{-b_1 a_1 \ln\ln 2} (\ln n)^{1/\alpha} n^{-(b_1 a_1 - \beta)},
\end{aligned}$$

and since $b_1 a_1 - \beta > 1$, we have $\sum_{n=1}^\infty K_1 e^{-b_1 a_1 \ln\ln 2} (\ln n)^{1/\alpha} n^{-(b_1 a_1 - \beta)} < \infty$. Hence, again the Borel–Cantelli lemma yields

$$\max_{x \in G_n} L(X_s + x, C_n) < a_1^{1/\alpha} g(2^{-n}) \tag{4.5}$$

for all but finitely many n's.

Choose γ so that $\gamma < \frac{\alpha - 1}{2}$. Let

$$\begin{aligned}
B_n = \bigcup_{k=1}^\infty \bigcup_{y_1, y_2} \Big\{ &|L(X_s + y_1, C_n) - L(X_s + y_2, C_n)| \\
&\geq (2^{-n})^{1-\frac{1+\gamma}{\alpha}} |y_1 - y_2|^\gamma (a_2 k \ln n)^{\frac{1+\gamma}{\alpha}} \Big\},
\end{aligned}$$

where y_1 and y_2 are lattice points in G_n that satisfy $y_1 - y_2 = \theta_n 2^{-k}$. Note that for each k there are 2^k such pairs of y_1 and y_2. It follows from Lemma 4.1 that

$$\mathbb{P}(B_n) = n^\beta (\ln n)^{\frac{1}{\alpha}} \sum_{k=1}^\infty 2^k K_2 e^{-b_2 a_2 k \ln n} \leq 3K_2 (\ln n)^{\frac{1}{\alpha}} n^{-(b_2 a_2 - \beta)},$$

where we have used the fact that

$$\sum_{k=1}^{\infty} 2^k n^{-b_2 a_2 k} = \sum_{k=1}^{\infty} \left(\frac{2}{n^{b_2 a_2}}\right)^k = \frac{\frac{2}{n^{b_2 a_2}}}{1 - \frac{2}{n^{b_2 a_2}}} = \frac{2}{n^{b_2 a_2} - 2} \leq 3n^{-b_2 a_2}$$

for all sufficiently large n. Hence, by taking a_2 so that $b_2 a_2 - \beta > 1$ we have by the Borel–Cantelli lemma

$$\mathbb{P}\Big(\limsup_n B_n\Big) = 0.$$

Now, suppose that $|y| < 2^{-\frac{n}{\alpha}} n^{\beta}$. Then, we can express y as $y = \lim_{k\to\infty} y_k$, where $y_0 = x$ and $y_k = x + \theta_n \sum_{j=1}^{k} \varepsilon_j 2^{-j}$. Hence, it follows from the triangular inequality that on the event $\Big\{\limsup_n B_n\Big\}^c$ and n sufficiently large,

$$\begin{aligned}
&|L(X_s + y, C_n) - L(X_s + x, C_n)| \\
&\quad \leq \sum_{k=1}^{\infty} |L(X_s + y_k, C_n) - L(X_s + y_{k-1}, C_n)| \\
&\quad \leq \sum_{k=1}^{\infty} (2^{-n})^{1-\frac{1+\gamma}{\alpha}} |y_k - y_{k-1}|^{\gamma} (a_2 k \ln n)^{\frac{1+\gamma}{\alpha}} \\
&\quad \leq \sum_{k=1}^{\infty} (2^{-n})^{1-\frac{1+\gamma}{\alpha}} \Big(2^{-\frac{n}{\alpha}} (\ln n)^{-\frac{1}{\alpha}} 2^{-k}\Big)^{\gamma} (a_2 k \ln n)^{\frac{1+\gamma}{\alpha}} \\
&\quad = (2^{-n})^{1-\frac{1}{\alpha}} (\ln n)^{\frac{1}{\alpha}} a_2^{\frac{1+\gamma}{\alpha}} \sum_{k=1}^{\infty} 2^{-k\gamma} k^{\frac{1+\gamma}{\alpha}} \leq c\, g(2^{-n})
\end{aligned} \tag{4.6}$$

for some finite constant $c > 0$. Hence, (4.4) follows from (4.5) and (4.6).

Next, we prove (4.3). For simplicity, we may and will assume that $I \subset [0, 1]$. Let $\mathcal{D}_n$ be a collection of 2^n non-overlapping intervals in [0, 1] of length $\frac{1}{2^n}$. Define $h(r) = r^{1-\frac{1}{\alpha}} (\ln r^{-1})^{\frac{1}{\alpha}}$. Let $\eta_n := 2^{-\frac{n}{\alpha}} n^{-\frac{1}{\alpha}}$ and define

$$H_n := \{x \in \mathbb{R} : |x| \leq n, \quad x = \eta_n p, \quad p \in \mathbb{Z}\}.$$

Note that the cardinality $\#(H_n)$ of H_n satisfies

$$\# H_n \leq \frac{2n}{\eta_n} = \frac{2n}{2^{-\frac{n}{\alpha}} n^{-\frac{1}{\alpha}}} = 2n^{1+\frac{1}{\alpha}} 2^{\frac{n}{\alpha}}.$$

It follows from Lemma 4.1 that

$$\begin{aligned}
&\mathbb{P}\left(\max_{x\in H_n} L(x,B) \geq a_3^{\frac{1}{\alpha}} h(2^{-n}) \text{ for some } B\in\mathcal{D}_n\right)\\
&\leq \mathbb{P}\left(\max_{x\in H_n} L(x,B) \geq (2^{-n})^{1-\frac{1}{\alpha}}(a_3 n\ln 2)^{\frac{1}{\alpha}} \text{ for some } B\in\mathcal{D}_n\right)\\
&\leq 2n^{1+\frac{1}{\alpha}}2^{\frac{n}{\alpha}}\times 2^n K_1 e^{-b_1 a_3 n\ln 2} = 2K_1 n^{1+\frac{1}{\alpha}}2^{-n(a_3 b_1-\frac{1}{\alpha})}.
\end{aligned}$$

Hence, by taking a_3 so that $a_3 b_1 - \frac{1}{\alpha} > 0$, we have

$$\max_{x\in H_n} L(x,B) < a_3^{\frac{1}{\alpha}} h(2^{-n}) \text{ for all } B\in\mathcal{D}_n \tag{4.7}$$

for all sufficiently large n.

Let

$$\begin{aligned}
D_n = \bigcup_{k=1}^{\infty}\bigcup_{y,y'}\Big\{ &\big|L(y,B)-L(y',B)\big|\\
&\geq (2^{-n})^{1-\frac{1+\gamma}{\alpha}}|y-y'|^{\gamma}(a_4 kn)^{\frac{1+\gamma}{\alpha}} \text{ for some } B\in\mathcal{D}_n\Big\},
\end{aligned}$$

where y and y' are lattice points in H_n that satisfy $y-y'=\eta_n 2^{-k}$. Hence, by Lemma 4.1, we have

$$\begin{aligned}
\mathbb{P}(D_n) &\leq 2n^{1+\frac{1}{\alpha}}2^{\frac{n}{\alpha}}\sum_{k=1}^{\infty}2^k e^{-b_2 a_4 kn} = 2n^{1+\frac{1}{\alpha}}2^{\frac{n}{\alpha}}\sum_{k=1}^{\infty}\left(\frac{2}{e^{b_2 a_4 n}}\right)^k\\
&\leq 2n^{1+\frac{1}{\alpha}}2^{\frac{n}{\alpha}}\frac{\frac{2}{e^{-b_2 a_4 n}}}{1-\frac{2}{e^{-b_2 a_4 n}}} = 2n^{1+\frac{1}{\alpha}}2^{\frac{n}{\alpha}}\frac{2}{e^{b_2 a_4 n}+2}\\
&\leq 6n^{1+\frac{1}{\alpha}}2^{\frac{n}{\alpha}}e^{-b_2 a_4 n} \leq 6n^{1+\frac{1}{\alpha}}e^{-n(b_2 a_4-\frac{\ln 2}{\alpha})}
\end{aligned}$$

for all sufficiently large n's. By taking a_4 so that $b_2 a_4 - \frac{\ln 2}{\alpha} > 0$, we see from the Borel–Cantelli lemma that $\mathbb{P}\big(\limsup_n D_n\big) = 0$.

By Lemma 4.2, we have

$$\mathbb{P}\Big(\sup_{0\leq t\leq 1}|X_t| = \infty\Big) = \lim_{n\to\infty}\mathbb{P}\Big(\sup_{0\leq t\leq 1}|X_t| > n\Big) \leq \lim_{n\to\infty} c\psi^*(1/n) = 0.$$

Hence, $\mathbb{P}(\sup_{0\leq t\leq 1}|X_t| < \infty) = 1$ and a.s. for each ω there exists n such that $\sup_{0\leq t\leq 1}|X_t| \leq n$. Hence, for n sufficiently large, if $|y|>n$, then $L(y,[0,1])=0$. For $|y|\leq n$, on the event $\big\{\limsup_n D_n\big\}^c$, we can express y as $y=\lim_{k\to\infty} y_k$ with

$y_0 = x$ as before. Hence, for any $B \in \mathcal{D}_n$ and $x \in H_n$, we have

$$\begin{aligned} |L(x, B) - L(y, B)| &= \sum_{k=1}^{\infty} |L(y_k, B) - L(y_{k-1}, B)| \\ &\le \sum_{k=1}^{\infty} (2^{-n})^{1-\frac{1+\gamma}{\alpha}} |y_k - y_{k-1}|^{\gamma} (a_4 kn)^{\frac{1+\gamma}{\alpha}} \\ &\le \sum_{k=1}^{\infty} (2^{-n})^{1-\frac{1+\gamma}{\alpha}} \left|2^{-\frac{n}{\alpha}} n^{-\frac{1}{\alpha}} 2^{-k}\right|^{\gamma} (a_4 kn)^{\frac{1+\gamma}{\alpha}} \\ &= a_4^{\frac{\gamma}{\alpha}} (2^{-n})^{1-\frac{1}{\alpha}} n^{\frac{1}{\alpha}} \sum_{k=1}^{\infty} 2^{-k\gamma} k^{\frac{1+\gamma}{\alpha}} \le c_2 h(2^{-n}) \end{aligned} \tag{4.8}$$

for some constant $c_2 > 0$. Now, (4.3) follows from (4.7) and (4.8). □

The following lemma is an analog of [22, Lemma 3.1].

Lemma 4.5 *Suppose that* $\psi \in \mathrm{WUSC}(\overline{\alpha}, \overline{\theta}, \overline{C})$ *and let* $\gamma < \frac{1}{\overline{\alpha}}$. *Then, there exists a constant* K *such that* $\mathbb{P}^x$*-a.s., for all* n *sufficiently large,* $X(I)$ *can be covered by* K *intervals of length* $2^{-n\gamma}$ *for all* $I \in \mathcal{C}_n$.

Proof It follows from Lemma 4.3 that for all n such that $2^{-n} \le \frac{1}{\overline{\theta}}$

$$\mathbb{P}^x\left(\sup_{0 \le s \le 2^{-n}} |X_s - x| \ge 2^{-n\gamma} \right) \le c2^{-n}(2^{-n\gamma})^{-\overline{\alpha}} = c2^{-n(1-\gamma\overline{\alpha})}.$$

Now, the conclusion of lemma follows from [23, Lemma 2.1]. □

Now, we are ready to prove Theorem 1.3 (ii).

Proof of Theorem 1.3 (ii) The proof is almost identical to the proof of [22, Theorem 1.1] using Theorem 4.4 and Lemma 4.5 instead of [22, Lemmas 3.1 and 3.2] and we briefly sketch main steps. For any Borel set $F \subset \mathbb{R}$, one can find a probability measure μ supported on F with $\mu(B) \le \mathrm{diam}(B)^{\dim_{\mathrm{H}} F - \varepsilon}$ for any B with $\mathrm{diam}(B) \le 1$. Define a measure λ supported on $\mathbb{R}^+$ as in [22, Equation (3.2)]. Then, using Theorem 4.4 and Lemma 4.5, one can argue that the measure λ satisfies $\lambda(B) \le \mathrm{diam}(B)^{1-\frac{1}{\alpha}+\gamma \dim_{\mathrm{H}} F - 2\varepsilon}$ for any $\gamma < 1/\alpha$ and all Borel sets B with sufficiently small diameter. This shows

$$\mathbb{P}^x\left(\dim_{\mathrm{H}} X^{-1}(F) \ge 1 - \frac{1}{\alpha} + \gamma \dim_{\mathrm{H}} F - 2\varepsilon \text{ for all compact Borel sets } F \right) = 1$$

and letting $\gamma \uparrow \frac{1}{\alpha}$, then $\varepsilon \downarrow 0$ establishes the claim. □

5 Examples

In this section, we provide some interesting examples to illustrate applications of the results of this paper. We recall from [6] that a function $f : I \to \mathbb{R}$ is said to be almost increasing with a factor $c \in (0, 1]$ if $cf(x) \le f(y)$ for all $x, y \in I$ and $x \le y$. A function $f : I \to \mathbb{R}$ is said to be almost decreasing with a factor $C \in [1, \infty)$ if $Cf(x) \ge f(y)$ for all $x, y \in I$ and $x \le y$. Finally, we recall the following characterizations for weakly scaling conditions.

Lemma 5.1 [6, Lemma 11] *We have* $\phi \in \mathrm{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$ *if and only if* $\phi(\theta) = \kappa(\theta)\theta^{\underline{\alpha}}$ *and* κ *is almost increasing on* $(\underline{\theta}, \infty)$ *with an oscillation factor* $\underline{c}$*. Similarly,* $\phi \in \mathrm{WUSC}(\overline{\alpha}, \overline{\theta}, \overline{C})$ *if and only if* $\phi(\theta) = \kappa(\theta)\theta^{\overline{\alpha}}$ *and* κ *is almost decreasing on* $(\overline{\theta}, \infty)$ *with an oscillation factor* $\overline{C}$*.*

1. Symmetric stable processes
   Let X^{SS} be a symmetric stable Lévy process in $\mathbb{R}$. The characteristic exponent of X is $\psi^{\mathrm{SS}}(\xi) = |\xi|^\alpha$, $\alpha \in (0, 2]$. When $\alpha = 2$, X^{SS} is a Brownian motion whose sample paths are continuous, which we exclude in this paper.
   If $\alpha \in (1, 2)$, then clearly $|\xi|^\alpha \in \mathrm{WUSC}(\alpha, 0, 1) \cap \mathrm{WLSC}(\alpha, 0, 1)$. Hence, (1.3) and (1.5) hold.
2. Relativistic stable processes
   Let X^{RS} be the relativistic stable process with mass m in $\mathbb{R}$. The characteristic exponent of X is given by

$$\psi^{\mathrm{RS}}(\xi) = (\xi^2 + m^{2/\alpha})^{\alpha/2} - m, \quad \xi \in \mathbb{R}^1, m > 0.$$

   Write

$$\psi^{\mathrm{RS}}(\xi) = |\xi|^\alpha \kappa_1(\xi), \quad \kappa_1(\xi) = \frac{(\xi^2 + m^{2/\alpha})^{\alpha/2} - m}{|\xi|^\alpha}.$$

   It is easy to check that

$$\lim_{\xi \to \infty} \kappa_1(\xi) = 1, \tag{5.1}$$

   and

$$\lim_{\xi \to 0} \frac{\kappa_1(\xi)}{|\xi|^{2-\alpha}} = \lim_{\xi \to 0} \frac{\left(m(\frac{|\xi|^2}{m^{2/\alpha}} + 1)^{2/\alpha} - m\right)}{|\xi|^2} = \frac{2}{\alpha m^{2/\alpha}}. \tag{5.2}$$

   If follows from (5.1) and (5.2), $\kappa_1(\xi)$ is almost increasing on $(0, \infty)$ and is almost decreasing on (θ, ∞) for some $\theta > 0$. This shows that $\psi^{\mathrm{RS}} \in \mathrm{WLSC}(\alpha, 0, \underline{c}) \cap \mathrm{WUSC}(\alpha, \theta, \overline{C}_1)$ from Lemma 5.1. Hence, when $\alpha \in (1, 2)$, (1.3) and (1.5) hold.
3. Truncated stable processes
   Let X^{TS} be the truncated stable Lévy process. The characteristic exponent of X^{TS}

is given by

$$\psi^{\mathrm{TS}}(\xi) = \int_{\{0<|y|\le 1\}} (1-\cos(y\xi))\frac{c(\alpha)}{|y|^{1+\alpha}}\mathrm{d}y,$$

where $c(\alpha)$ is a constant so that $\int_{\mathbb{R}\setminus\{0\}}(1-\cos(y\xi))\frac{c(\alpha)}{|y|^{1+\alpha}}\mathrm{d}y = 1$. By the change of variable $y = \frac{x}{|\xi|}$, we observe that

$$\psi^{\mathrm{TS}}(\xi) = c(\alpha)|\xi|^{\alpha}\int_{\{0<|x|\le|\xi|\}} \frac{1-\cos(\frac{\xi x}{|\xi|})}{|x|^{1+\alpha}}\mathrm{d}x.$$

Hence, we have $\psi^{\mathrm{TS}}(\xi) \sim |\xi|^{\alpha}$ as $\xi \to \infty$. Since $1-\cos(\frac{\xi x}{|\xi|}) \sim |x|^2$ as $|x| \to 0$, we observe that $\psi^{\mathrm{TS}}(\xi) \sim c(\alpha)|\xi|^2$ as $\xi \to 0$. Write $\psi^{\mathrm{TS}}(\xi)$ as

$$\psi^{\mathrm{TS}}(\xi) = |\xi|^{\alpha}\kappa_2(\xi).$$

Then, we observe that

$$\lim_{\xi\to 0}\kappa_2(\xi) = \lim_{\xi\to 0}\frac{\psi^{\mathrm{TS}}(\xi)}{|\xi|^{\alpha}} = 0 \text{ and } \lim_{\xi\to\infty}\kappa_2(\xi) = 1.$$

Hence, we see that $\kappa_2(\xi)$ is almost increasing on $(0,\infty)$ and is almost decreasing on (θ,∞) for some $\theta > 0$. This shows that $\psi^{\mathrm{TS}}(\xi) \in \mathrm{WLSC}(\alpha, 0, \underline{c}) \cap \mathrm{WUSC}(\alpha, \theta, \overline{C}_1)$. Hence, we conclude that (1.3) and (1.5) hold.

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