The Lebesgue Theory of Measure and Integration

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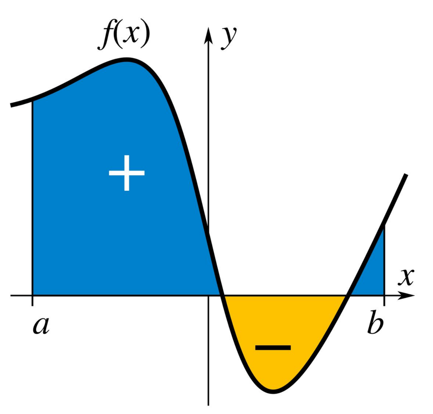
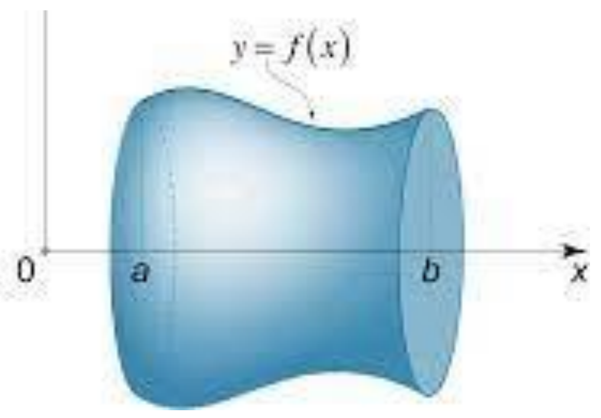
What's an Integral?

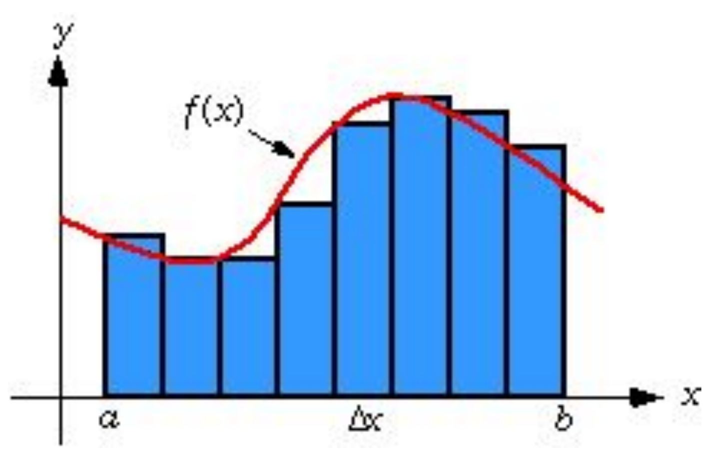
- **Integral** = an operation applied to a function, can often be interpreted as the area under the curve.
- Integrals are **vital** to modern **math, physics, and engineering**. Their applications include:
  - Finding volumes of complex shapes
  - Computing probabilities
  - Calculating bulk quantities like work or energy
  - Solving differential equations
- Integration is related to **length, area, & volume!**





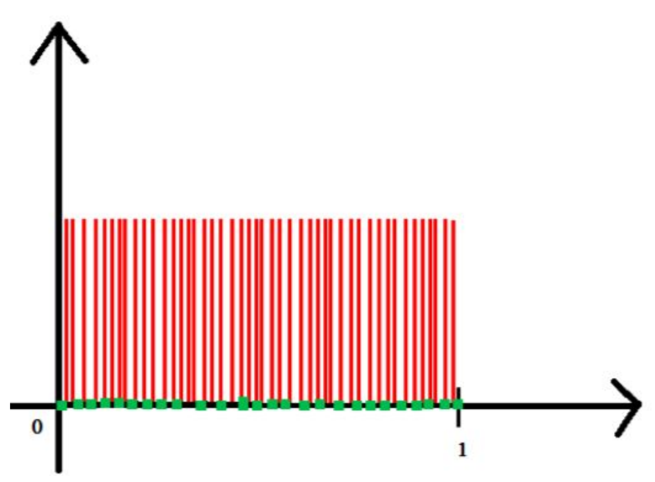
The Riemann Integral

- The Riemann integral is the standard method of defining an integral in Calculus classes.
- Riemann integration = approximating area under curve by vertical rectangles



The Riemann integral is easier to define + apply, but it has a lot of theoretical issues!

- **Example 1:** You can find a sequence of Riemann integrable functions that converges to a non-Riemann integrable function
- **Example 2:** The Dirichlet function $\chi_{\mathbb{Q}}(x)$
  - 1 if x is rational
  - 0 if x is irrational
  - *Very* discontinuous, has no clear area under curve



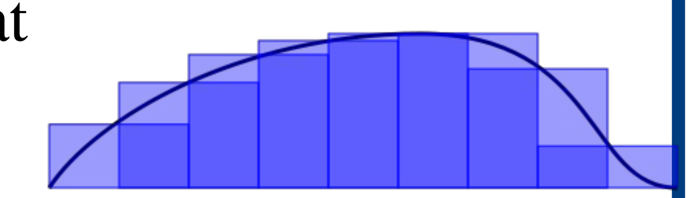
The Problem of Measure

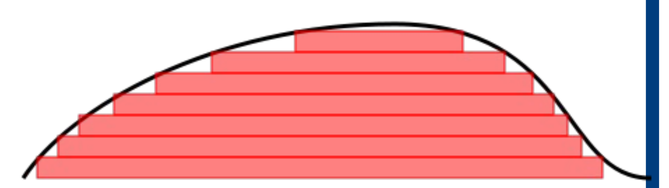
- Integration motivates us to ask: **how do we measure the "size" of sets** in Euclidean space? (e.g. $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$…)
  - Can we measure the size of every set?
- In the **physical world**, this is easy: we use a ruler.
- In **math**, there are sets that can't be measured.
- **Example:** Banach-Tarski paradox: given a 3D sphere, you can **break it into pieces** and put them together in such a way that creates **a second sphere**.
  - These pieces are *non-measurable*, because if we could measure them, their volumes wouldn't add up.

- The Lebesgue philosophy excludes non-measurable sets, and focuses on *measurable* sets: sets whose size can be approximated by simple sets (e.g. open sets)
- **The Lebesgue measure**: A function which gives the measure (area, volume) of a set in Euclidean space.
- If we can approximate the size of a set E by an open set, then the Lebesgue measure of E is the size of the smallest open set that contains E.

The Lebesgue Integral

- The **Lebesgue integral** seeks to address the problems with the Riemann integral, and builds on measure theory and the Lebesgue measure to do so.
- **Riemann** integration partitions the **domain**, **Lebesgue** integration effectively partitions the **range**
- **Simple functions**: functions that attain finitely many values, where the sets on which each value is attained are measurable and have finite measure





The Lebesgue Integral (cont.)

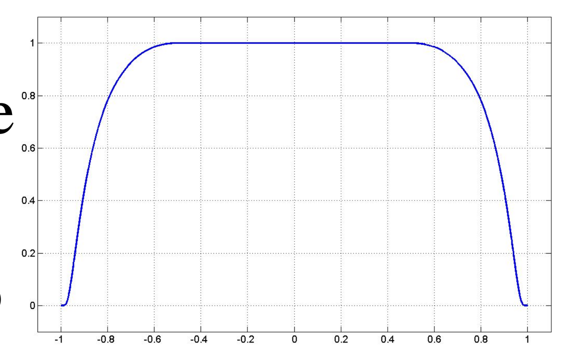
- We can approximate "measurable" functions (roughly, functions whose range is measurable) by simple functions by **partitioning the range.**
- *Therefore, we can define the Lebesgue integral by starting with simple functions, and then approximating more general functions by simple functions.*

Constructing the Lebesgue Integral

- **Step 1:** Simple functions
  - The Lebesgue integral is the sum of the values the function attains times the (Lebesgue) measure of the sets on which each value is attained.

$$s(x) = \sum_{k=1}^{N} c_k \chi_{E_k} \implies \int s(x)dx = \sum_{k=1}^{N} c_k m(E_k)$$

- **Step 2:** Bounded functions supported on a set of finite measure
  - **Support** of a function = the set on which the function is nonzero
  - These can be approximated by simple functions!



- **Step 3:** Non-negative functions
  - For a non-negative function *f(x)*, consider the set of bounded functions *g(x)* supported on a set of finite measure with $g(x) < f(x)$.
  - Then take the integral of each *g(x)*. The Lebesgue integral of *f(x)* is the largest such integral.
  - Essentially, we approximate *f(x)* from below by bounded functions supported on a set of finite measure

$$\int f = \sup \int g \qquad g(x) \leq f(x)$$

- **Step 4:** General functions
  - Here, we decompose a function into the difference of two non-negative functions and use the previous step.